109. Band Heft 3, September 2007 D 20577

Jahresbericht

der Deutschen Mathematiker-Vereinigung

3 – 2007

Herausgegeben von K. Hulek
unter Mitwirkung von
U. Gather, H.-Ch. Grunau, H. Lange,
J. Rambau, A. Schied, Th. Sonar

Teubner

Jahresbericht
der Deutschen Mathematiker-Vereinigung

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Verlag:
B. G. Teubner Verlag | GWV Fachverlage GmbH
Abraham-Lincoln-Straße 46
65189 Wiesbaden
http://www.teubner.de
http://www.gwv-fachverlage.de

Geschäftsführer: Andreas Kösters,
Dr. Ralf Birkelbach,
Albrecht F. Schirmacher
Gesamtleitung Anzeigen: Thomas Werner
Gesamtleitung Produktion: Ingo Eichel
Gesamtleitung Vertrieb: Gabriel Göttlinger

Marketing/Sonderdrucke:
Eva Brechtel-Wahl
Telefon: (06 11) 78 78-3 79, Fax: (06 11) 78 78-4 39, E-Mail: eva.brechtel-wahl@gwv-fachverlage.de

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(Änderungen von Adressen und Bankverbindung, Rückfragen zu Rechnungen oder Mahnung)
VVA-Zeitschriftenservice, Abt. D6F6 / Jahresbericht der Deutschen Mathematiker-Vereinigung, Postfach 7777, 33310 Gütersloh
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www.gwv-anzeigen.de

Es gilt die Preisliste vom 01.01.2007.

Bezugsbedingungen:
Die Zeitschrift erscheint 4mal jährlich zum Jahresabonnementpreis von € 118,– inkl. Versandkosten. Der Bezug von Einzelheften ist nicht möglich. Schriftliche Kündigung des Abonnements spätestens sechs Wochen vor Ablauf des Bezugsjahres.
Für persönliche Mitglieder der DMV, die den Jahresbericht zu beziehen wünschen, ist der zwischen DMV und Verlag vereinbarte Bezugspreis maßgebend, der im Rahmen des Mitgliedsbeitrags erhoben wird.

Satz: Fotosatz Behrens, D-68723 Oftersheim
Druck: Wilhelm & Adam, Heusenstamm

ISSN 0012-0456

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Jahresbericht der Deutschen Mathematiker-Vereinigung, 109. Bd. 2007, Nr. 3

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Nachdruck der Bände 41 bis 67 liefert: Swets & Zeitlinger, Heereweg 347 b, POB 810, NL-2160 SZ Lisse/Holland

Jahresbericht der Deutschen Mathematiker-Vereinigung, 109. Bd. 2007, Nr. 3

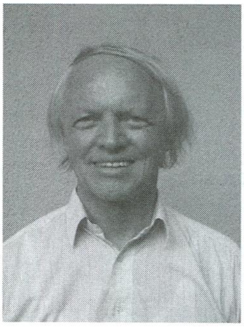
Vorwort

Im Zentrum dieses Hefts stehen zwei ausführliche Übersichtsartikel. L. Rüschendorf berichtet über neue Entwicklungen zum sogenannten Monge-Kantorovich Problem, bei dem es um den optimalen Massentransport zwischen zwei Gebieten geht. Hier treffen Probleme der Analysis und der Geometrie aufeinander, wobei insbesondere die Theorie der partiellen Differentialgleichungen, Riemannsche Geometrie und Variationsprobleme eine Rolle spielen. In dem vorliegenden Aufsatz werden vor allem die probalitistischen Aspekte des Problems eingehend dargestellt. An dieser Stelle sei noch erwähnt, dass Kantorovich zusammen mit Koop im Jahre 1975 den Nobelgedächtnispreis für Wirtschaftswissenschaften erhalten hat.

Der zweite Übersichtsartikel dieser Ausgabe stammt von G. Schneider und H. Uecker und behandelt das Verhalten von Lichtimpulsen in dispersiven Medien wie etwa Glasfaserkabeln. Dies ist eine Frage von grundlegender Bedeutung für die optische Datenverarbeitung und insbesondere den Transport von Informationen, eine der wesentlichen Grundlagen unserer Informationsgesellschaft. Mathematisch gesehen handelt es sich dabei um die Untersuchung nichtlinearer Wellengleichungen, wobei insbesondere die nicht-lineare Schrödingergleichung eine wichtige Rolle spielt.

Das Heft wird wie immer durch aktuelle Buchbesprechungen ergänzt.

K. Hulek



Monge-Kantorovich Transportation Problem and Optimal Couplings

Ludger Rüschendorf

Abstract

- Mathematics Subject Classification: 60 E 15, 90 C 46, 65 K 10, 39 B 72
- Keywords and Phrases: Optimal coupling, minimal metric, duality theory, polar factorization, c-convexity, stochastic equation, recursive algorithm

The Monge-Kantorovich mass-transportation problem has been shown in recent years to be fundamental for various basic problems in analysis and geometry. In this paper we describe some of the historical developments of this probem and some of the basic results. In particular we emphasize the probabilistic aspects and contributions to this subject and its relevance for various classical and recent developments in probability theory ranging from probability metrics and functional inequalities over estimates for risk measures to the analysis of algorithms. The paper is based on a lecture of the author delivered at the MSRI meeting on mass transportation problems in November 2005 in Berkeley.

Eingegangen: 23. 10. 2006

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1 Introduction

The classical mass transportation problem of Monge and its version of Kantorovich has found a lot of recent interest because of its importance for several problems in nonlinear PDEs, Riemannian geometry, variational problems and for several interesting inequalities and concentration results, see in particular the recent excellent presentations of Ambrosio (2003), Villani (2003, 2006) and Ambrosio, Gigli, and Savaré (2005). In this paper we survey some of the probabilistic developments of the transportation problem and the related optimal coupling problem and its applications. The probabilistic development of the subject was mainly concentrated on the Kantorovich formulation of the problem which turned out to be also instrumental for the Monge formulation and its applications in analysis.

The probabilistic interest in this topic was essentially connected with some naturally defined minimal metrics on the space of probability measures which are defined via optimal coupling properties. In particular to mention are the minimal ℓ_p-metrics, the Kantorovich-Rubinstein theorem and others. Much of this development and many probabilistic applications are discussed in Rachev and Rüschendorf (1998a,b).

After the introduction of the connections between optimal couplings and mass transportation we discuss in section 3 the development of the basic duality theory which gives the clue to many of the optimal coupling results. We then present in section 4 as consequences some of the main results for optimal L^2-couplings (the classical L^2-distance) and also for general coupling functions. This includes in particular the important characterizations of optimal transportation plans based on generalized convexity notions (c-convexity, c-subgradients, c-cyclical monotonicity). At this point roughly around 1990 the development of this subject in analysis began. Here this survey tries to relate the various historical sources and to describe the probabilistic contributions. We also describe some of the more concrete probabilistic applications and developments as e.g. to the optimal coupling of normal or discrete distributions or to obtain bounds for the risk of portfolios arising from positive dependence. In the final part of this paper we discuss a recently introduced modification of the minimal ℓ_s-metric and its application in the analysis of recursive algorithms of divide-and-conquer type.

2 The mass transportation problem

In 1942 Kantorovich introduced the problem of *optimal mass transport* in the following form:

$$\int c(x,y)d\mu(x,y) = \inf_{\mu\in M(P_1,P_2)} =: \widehat{\mu}_c(P_1,P_2), \tag{2.1}$$

where $c : U_1 \times U_2 \to \mathbb{R}$ is a measurable real cost function, $P_i \in M^1(U_i)$ are probability measures on U_i and

$$M(P_1,P_2) = \{\mu \in M^1(U_1 \times U_2); \mu^{\pi_i} = P_i, \quad i = 1,2\} \tag{2.2}$$

is the class of all probability measures on $U_1 \times U_2$ with marginals P_1, P_2. Here π_i are the projections on the i-th components and μ^{π_i} is the image of μ under π_i. $\widehat{\mu}_c$ is called the *Monge-Kantorovich functional.* In terms of random variables on a non-atomic probability space $(\Omega, \mathfrak{A}, P)$ problem (2.1) is equivalent with the problem to find an *optimal coupling* of P_1, P_2 w.r.t. the coupling function c, i.e.

$$Ec(X_1, X_2) = \inf \tag{2.3}$$

over all couplings X_1, X_2 of P_1, P_2, i.e. such that $P^{X_i} = P_i$, $i = 1, 2$. Any $\mu \in M(P_1, P_2)$ describes a transference (transportation) plan for the mass distribution P_1 to P_2 or equivalently the joint distribution of a pair of (X_1, X_2) of couplings of P_1, P_2. Using conditional distributions we obtain for $\mu \in M(P_1, P_2)$

$$\int c(x,y)d\mu(x,y) = \int \left(\int c(x,y)\mu(dy|x) \right) P_1(dx). \tag{2.4}$$

Any mass at point x is transported to y according to $\mu(dy|x)$ and thus $\int c(x,y)d\mu(x,y)$ denotes the total cost of transportation using this plan. In the optimal coupling problem (2.3) c is understood as a distance (dissimilarity) and it is a natural problem to find an coupling (X_1, X_2) of P_1, P_2 with minimal expected dissimilarity.

A subclass of all transport plans are deterministic transport plans of the form $\mu(\cdot \mid x) = \varepsilon_{\phi(x)}$, where ϕ is a function which transports P_1 to P_2, i.e. $P_1^{\phi} = P_2$. The additional restriction is that no mass is allowed to be split. Denoting by

$$S(P_1, P_2) = \{\phi : U_1 \to U_2, \phi \text{ measurable}, P_1^{\phi} = P_2\} \tag{2.5}$$

the set of all deterministic transport plans one obtains the corresponding *Monge transportation problem*

$$\int c(x, \phi(x))dP_1(x) = \inf_{\phi \in S(P_1, P_2)} \tag{2.6}$$

resp. the deterministic coupling problem

$$Ec(X_1, \phi(X_1)) = \inf_{\phi \in S(P_1, P_2)}. \tag{2.7}$$

This problem was introduced in 1781 by Monge for the special case that $U_i \subset \mathbb{R}^3$ are two bounded domains with volume measures P_i and $c(x,y) = \|x - y\|$ is the Euclidean distance. Monge detected that optimal transport should go along straight lines which are orthogonal to a family of surfaces (formally worked out by Appel (1887)). Also he found the no-crossing rule of optimal transport rays.

From the probabilistic point of view the Kantorovich formulation of the transport problem is more 'natural' than the Monge formulation. Similar extensions of deterministic optimization problems are quite often to find in probability and statistics as e.g. the transition from deterministic decision rules (like deterministic tests and estimators) to randomized decision rules (like randomized tests and estimators) is a classical example from the early period of statistics. Kantorovich obviously was not aware of the Monge problem when he formulated his transport problem in 1942. In 1948 he wrote a short note of three pages where he made the connection to the Monge problem and stated that in case (2.1) has a deterministic solution ϕ, then ϕ is also a solution of the Monge problem, i.e., the Kantorovich problem is a relaxation of the Monge problem. In fact Kantorovich's problem was one of the earliest infinite dimensional linear programming problems considered. In 1975 Kantorovich got together with Koopman the Nobel price in economics for his development of linear programming and the application to mathematical economics.

The Kantorovich problem and some variants have been developed in the probabilistic literature since the mid seventies. For various kinds of optimization problems they have been established as a basic and natural tool. A detailed exposition with many applications of these developments is given in Rachev and Rüschendorf (1998a,b). Starting with the late eighties, early nineties, important connections of the transportation problem with problems from analysis and geometry, partial differential equations, fluid mechanics, general curvature theory, variational problems, geometric and functional inequalities like isoperimetric and concentration inequalities, gradient flows in metric spaces, and many others have been detected. This lead to a very active and wide ranged research area. This line of research is excellently described and developed in the books of Ambrosio (2003), Villani (2003, 2006), and Ambrosio, Gigli, and Savaré (2005).

In the following we review some of the history of the probabilistic development of the transport problem, put it into line with the developments described above in analysis. In this way we obtain e.g. a new extension of Brenier's polar factorization result. Finally, we point out to some of the more recent applications in various areas of probabilistic analysis.

3 Duality theory and optimal couplings

In this section we describe developments of the duality theory which is the main tool and the basis for determining optimal couplings and transport plans. In some more recent work starting with McCann (1995) and Gangbo and McCann (1996) more direct methods have been developed to determine optimal transport plans. We begin this section with stating some of the classical results on minimal probability metrics which stand at the beginning of optimal transportation problems.

3.1 Minimal probability metrics

a) Minimal ℓ_1-metric. Let (U,d) be a separable metric space and $P_1, P_2 \in M^1(U)$ be probability measures on U with its Borel σ-Algebra. We denote the *minimal ℓ_1-metric on $M^1(U)$ by*

$$\ell_1(P_1,P_2) := \inf\left\{\int d(x,y)d\mu(x,y); \quad \mu \in M(P_1,P_2)\right\} \tag{3.1}$$

i.e. ℓ_1 is the minimal version of the usual L_1-metric

$$L_1(X,Y) = Ed(X,Y) \tag{3.2}$$

of random variables X, Y in U and is identical to the solution of the transportation problem with cost function $c = d$.

The *Lipschitz metric* μ_L is defined by

$$\mu_L(P_1, P_2) = \sup\left\{ \int f d(P_1 - P_2);\ \operatorname{Lip} f \leq 1 \right\}. \tag{3.3}$$

Kantorovich-Rubinstein Theorem: The minimal ℓ_1-metric is identical to the Lipschitz metric, i.e., for all $P_1, P_2 \in M^1(U)$ holds

$$\ell_1(P_1, P_2) = \mu_L(P_1, P_2). \tag{3.4}$$

This result was proved by Kantorovich and Rubinstein (1957) in the case of compact metric spaces and then extended by de Acosta (1982), Dudley (1976), Fernique (1981), Levin (1975), and Kellerer (1984a).

In the case of the real line $U = \mathbb{R}^1$ and $d(x, y) = |x - y|$ one gets the explicit expression

$$\begin{aligned} \ell_1(P_1, P_2) &= \int_0^1 |F_1^{-1}(u) - F_2^{-1}(u)| du \\ &= \int |F_1(x) - F_2(x)| dx \end{aligned} \tag{3.5}$$

where F_i are the distribution functions of P_i. In this case the results go back to early work of Gini (1914), Salvemini (1949), and Dall'Aglio (1956) (even for the case $c(x, y) = |x - y|^\alpha$) Vallander (1973) and Szulga (1978).

Fréchet (1940) was the first to note formally the metric properties of ℓ_1 in general metric spaces, Hoeffding (1940) gave a formula for ℓ_2 in the real case and Vasershtein (1969) 'introduced' ℓ_1 again in his paper on Markov processes. Dobrushin (1970) was the first to call ℓ_1 Wasserstein metric (Wasserstein the English transcripted version of Vasershtein).

b) Total variation metric. Let V denote the total variation metric on U and let

$$\sigma(X, Y) := P(X \neq Y) \tag{3.6}$$

denote the compound probability metric on the space of random variables, then

$$V(P_1, P_2) = \widehat{\sigma}(P_1, P_2) = \inf\{\sigma(X, Y); X \stackrel{d}{=} P_1, Y \stackrel{d}{=} P_2\}. \tag{3.7}$$

This result is due to Dobrushin (1970). It is a basic result to many of the optimal coupling results in probability theory which extend the classical paper of Doeblin (1938) giving a coupling proof of the limit theorem for Markov chains.

c) Prohorov metric. A similar result holds true for the Prohorov metric π on $M^1(U)$ which is the classical metric for the weak convergence topology. Strassen (1965) proved that π is the minimal metric of the Ky Fan metric K, i.e.

$$\pi = \widehat{K}. \tag{3.8}$$

In all three cases the transition to the minimal metric yields a change of the topology. This has important applications, e.g. to Skorohod type results (relation between weak and strong convergence) in the proof of central limit theorems, in matching theory and in robustness results and many others. More generally for any compound probability metric $\mu(X, Y)$ the corresponding minimal metric $\widehat{\mu}$ is defined by

$$\widehat{\mu}(P_1, P_2) = \inf\{\mu(X, Y); X \stackrel{d}{=} P_1, Y \stackrel{d}{=} P_2\}. \tag{3.9}$$

Zolotarev (1976) used this principle for the construction of several examples of ideal metrics. The interesting question of characterization of the minimal metrics with ζ-structure, i.e. which have a sup-representation similar to (3.3), is so far only partially answered.

3.2 Monge-Kantorovich duality theory

The duality theory for the transportation problem began with Kantorovich's 1942 result which stated equivalence of (2.4) with a dual problem for the case of compact metric spaces and continuous cost functions $c(x, y)$. The proof however worked only for the case where $c(x, y) = d(x, y)$ is a metric on $U = U_1 = U_2$. For the metric case the MK-transportation problem is equivalent to the mass transfer problem where for $P_1, P_2 \in M^1(U)$ the class of transport plans $M(P_1, P_2)$ is replaced by the class of *mass transference plans*

$$M = M(P_1 - P_2) = \{\gamma \in M(U \times U);\ \gamma^{\pi 1} - \gamma^{\pi 2} = P_1 - P_2\}, \tag{3.10}$$

i.e. all transport plans with fixed difference of the marginals. With respect to this class it is allowed to transfer a mass point of x to y via some route $x = x_1, x_2, \dots, x_n = y$ such that the cost $c(x, y)$ is replaced by the cost $\sum_{i=1}^{n} c(x_i, x_{i+1})$. The basic result is an extension of the Kantorovich-Rubinstein theorem of the form

$$\begin{aligned} \mathring{\mu}_c(P_1, P_2) &= \inf\left\{ \int c(x, y) d\mu(x, y); \mu \in M(P_1 - P_2)\right\} \\ &= \sup\left\{ \int_U f d(P_1 - P_2); f(x) - f(y) \le c(x, y)\right\}. \end{aligned} \tag{3.11}$$

After the Kantorovich-Rubinstein (1957) paper this kind of duality theorems for the mass transfer problem was intensively discussed in the Russian probability school in particular by Levin (1975) and Levin and Milyutin (1979). Also the papers of de Acosta (1982), Dudley (1976), Fernique (1981), and Rachev and Shortt (1990) concerned the Kantorovich-Rubinstein functional $\mathring{\mu}_c$. It coincides with the MK-functional $\widehat{\mu}_c$ only if c is a metric (see Neveu and Dudley (1980)). An important role in this development is played by the Lipschitz norm in (3.11) (see Fortet and Mourier (1953)) and by related approximation arguments. A rather complete duality theory of the KR-functional $\mathring{\mu}_c$ has been developed by Levin (see corresponding references and presentation in Rachev and Rüschendorf (1998a,b)).

The MK-problem with fixed marginals can be considered also on n-fold products of probability spaces $(U_i, \mathfrak{A}_i, P_i)$. Let $h : \prod_{i=1}^n U_i \to \mathbb{R}$ and $M = M(P_1, \dots, P_n)$ be the set of all transport plans, i.e. measures with marginals P_i then we define

$$\begin{aligned} S(h) &= \sup\left\{\int h d\mu; \mu \in M\right\} \\ I(h) &= \inf\left\{\sum_{i=1}^n \int f_i dP_i; h \le \oplus f_i, f_i \in L^2(P_i)\right\}, \end{aligned} \tag{3.12}$$

where $\oplus f_i(x) = \sum_{i=1}^n f_i(x_i)$. We say that *duality holds* if

$$S(h) = I(h). \tag{3.13}$$

Here the equivalent problem of maximizing the gain (profit) is considered which transfers to the problem of minimizing the cost by switching to $c = -h$.

For the proof of duality theorems of MK-type several strategies have been developed. One approach is to establish via Hahn-Banach and Riesz-type results in the first step the equality

$$\widetilde{S}(h) = I(h) \tag{3.14}$$

where $\widetilde{S}(h)$ is the supremum problem where the measures with fixed marginals are relaxed to the finite additive measures $\mathrm{ba}(P_1, \dots, P_n)$. In the second step conditions on h are identified (Riesz-type results) which ensure that $S(h) = \widetilde{S}(h)$. This approach was followed in Rüschendorf (1979–1981) (in the following abbreviated by Rü) and Gaffke and Rü (1981) (without being aware at that time of the MK-problem in the Russian school). Motivated by this development Kellerer (1984b) followed a different route to obtain more general results. Starting from the duality for simple cases he investigated in detail continuity properties of the functionals S, I which allowed him by Choquet's capacity theorem to obtain very general duality results. Rachev (1985) extended approximation arguments as used in the KR-case to some instances of the MK-problem. Levin (since 1984) established some techniques which allowed him to prove reduction results from the MK-problem to the KR-problem.

Here is a list of some of the basic duality results for the MK-duality problem. The spaces U_i are assumed to be Hausdorff and the measures are restricted to the class of tight measures.

Theorem 3.1 (Duality Theorem.) *a) Duality holds on the class of all lower-majorized product-measurable functions:*

$$\mathcal{L}_m(U) = \{h \in \mathcal{L}(\mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_n); \exists f_i \in L^1(P_i); h \ge \oplus_{i=1}^n f_i\}. \tag{3.15}$$

b) Duality holds on $\overline{F}(U)$, (3.16)

where $F(U)$ *are the upper semicontinuous functions and closure is w.r.t.* $I(|f - g|)$.

Duality also holds on $\overline{G}_m(U)$, (3.17)
the closure of the lower majorized lower semicontinuous functions.

c) Existence of an optimal measure on $\overline{F}$ for the S-functional. (3.18)

d) *Existence of minimal functions* (f_i) on $\mathcal{L}_m(U)$ for the I-functional. (3.19)

e) *For* $P \in M(P_1, P_2), f_i \in \mathcal{L}^1(P_i), \oplus f_i \geq h$ *holds:*

$$P, (f_i) \text{ are solutions for } S, I \iff h = \oplus_{i=1}^n f_i \ [P] \tag{3.20}$$

Remarks 3.2 a) The duality and existence results were proved in Rü (1981) essentially for the bounded product measurable case and in Kellerer (1984b) for the general case. The conditions in the results are sharp, i.e. there exist counter-examples of the duality and existence results, e.g. on $G(U)$ or without lower boundedness in d).

b) Condition (3.20) characterizes optimal transport plans P under the existence condition e.g. for $h \in \mathcal{L}_m(U)$. The sufficiency part does not need any conditions, i.e. the r.h.s. of (3.20) implies optimality of P and (f_i).

c) In Kellerer (1984b) a simple example is given where $h \in C(U \times U)$, $S(h) = I(h)$, $P_1 = P_2$, but h does not allow a representation of the form $h = f_1 \oplus f_2\, [P]$ with $f_i \in L^1(P_i), P \in M(P_1, P_2)$.

d) The existence of solutions of the dual problem is closely connected with the following *closedness problem:* Let $P \in M(P_1, P_2)$, $s \geq 0$ and consider

$$\begin{aligned} F_s &= L^s(P_1) \oplus L^s(P_2) \\ &= \{f = f(x, y) = g(x) + h(y); g \in L^s(P_1), h \in L^s(P_2)\} \end{aligned} \tag{3.21}$$

When is F_s closed in $L^s(P)$? In general closedness does not hold true (see Rü and Thomsen (1993)). Several partial results are known, e.g. in case $s = 0$ and $P \ll P_1 \otimes P_2$ any element $\Phi \in \overline{F}_0$, the closure w.r.t. $L^0(P)$, has a representation of the form

$$\Phi(x, y) = f(x) + g(y) \ [P] \tag{3.22}$$

but in general f, g cannot be chosen measurable. Several positive results are established. (3.22) is sufficient for proving the existence of a general version of Schrödinger bridges and the positive results also allow to give an extension of the Kolmogorov representation result for continuous functions of n variables by a superposition of functions of one variable to the case of locally bounded measurable functions with equality holding a.s. (see Rü and Thomsen (1997)).

Immediately after establishing the duality results (3.15)–(3.20) some interesting consequences were established in the early eighties in particular sharpening some classical bounds. Here are some examples:

a) **Sharpness of Fréchet-bounds:** For $A_i \in \mathfrak{A}_i$ holds

$$\sup\{P(A_1 \times \ldots \times A_n); P \in M(P_1, \ldots, P_n)\} = \min\{P_i(A_i); 1 \leq i \leq n\} \tag{3.23}$$

$$\inf\{P(A_1 \times \ldots \times A_n); P \in M(P_1, \ldots, P_n)\} = \left(\sum_{i=1}^n P_i(A_i) - (n-1)\right)_+. \tag{3.24}$$

These are classical bounds in probability theory. They could be shown to be sharp by calculating the dual problem explicitly (see Rü (1981a)).

b) **Hölder and Jensen inequality:** For $\alpha_i > 0, \sum 1/\alpha_i = 1, X_i \geq 0$ the Hölder inequality

$$E\prod_{i=1}^{n} X_i \leq \prod_{i=1}^{n} \|X_i\|_{\alpha_i} \tag{3.25}$$

is an optimal upper bound in the class of distributions with given α_i-th moments of X_i. One can improve this bound by

$$E\prod_{i=1}^{n} X_i \leq E\prod_{i=1}^{n} F_i^{-1}(U), \tag{3.26}$$

where U is uniform on $(0, 1)$, F_i the distribution functions of P_i. This bound is sharp in the class of all distributions with marginals P_i (with distribution functions F_i).

Similarly the Jensen inequality

$$E\varphi(X) \underset{(\geq)}{\leq} \varphi(EX), \qquad \varphi \text{ convex (concave)} \tag{3.27}$$

is sharp in the class of all distributions with given expectation. For large classes of convex functions one can improve the bounds. E.g. for $\varphi(x) = \max x_i - \min x_i$, the span of x, holds

$$E \text{ span}(X_i) \geq E \text{ span}(F_i^{-1}(U)) \tag{3.28}$$

which is sharp in the class of distributions with given marginals.

c) **Sharp bounds for the sum:** For $P_i \in M^1(\mathbb{R}^1)$ with distribution functions F_i holds

$$\begin{aligned} \sup\{P(X_1 + X_2 \leq t); X_i \sim P_i\} &= F_1 \wedge F_2(t) \\ &= \inf\{F_1(u) + F_2(t-u); u \in \mathbb{R}^1\}, \end{aligned} \tag{3.29}$$

$F_1 \wedge F_2$ is the infimal convolution of the distribution functions F_i of P_i. Similarly,

$$\inf\{P(X_1 + X_2 \leq t); X_i \sim P_i\} = (F_1 \vee F_2(t) - 1)_+, \tag{3.30}$$

where $F_1 \vee F_2$ is the supremal convolution.

This problem of sharp bounds for the distribution of the sum was solved independently by Makarov (1981) and Rü (1982). In Rü (1982) the proof was based on the duality theorem. This result has found recently great interest in risk theory since it allows to derive sharp bounds for the *'value at risk'* measure in a portfolio caused by dependence of the components. There are a sequence of recent papers using the duality result in order to establish effective extensions of the bounds for more than two random variables and thus to obtain effective bounds for the risk in portfolios.

d) For Δ-monotone and for quasi monotone functions $h : \mathbb{R}^n \to \mathbb{R}^1$ as e.g. $h(x) = \phi(x_1 + \ldots + x_n)$, ϕ convex, sharp bounds were established

$$\sup\{Eh(X); X_i \sim P_i\} = Eh(F_1^{-1}(U); \ldots, F_n^{-1}(U)) \tag{3.31}$$

(see Tchen (1980), Rü (1980, 1983)).

The duality theorem (3.15)–(3.20) was established for tight measures on a Hausdorff space which corresponds roughly to the case of complete separable metric spaces. A natural question is in what generality does the duality theorem hold true? We consider the case $n = 2$ and probability spaces $(U_i, \mathfrak{A}_i, P_i)$ and define:

(D) holds if $S(h) = I(h)$ for all $h \in B(U_1 \times U_2, \mathfrak{A}_1 \otimes \mathfrak{A}_2)$ (or $h \in \mathcal{L}_m(U_1 \times U_2)$).

Remind that $P \in M^1(\Omega, \mathfrak{A})$ is called *perfect* if for all $f \in \mathcal{L}(\mathfrak{A})$ $\exists$ a Borel set $B \subset f(\Omega)$ such that

$$P(f^{-1}(B)) = 1. \tag{3.32}$$

This notion was introduced by Kolmogorov and is instrumental for various measure theoretic constructions like conditional probability measures (see Ramachandran (1979a,b)). The following general duality theorem holds if one of the underlying marginal measures is perfect.

Theorem 3.3 (Perfectness and duality.) *(Ramachandran and Rü (1995)) If P_2 is perfect, then $(U_2, \mathfrak{A}_2, P_2)$ is* a duality space, *i.e. (D) holds for any further probability space $(U_1, \mathfrak{A}_1, P_1)$.*

The proof starts with the case $U_i = [0, 1]$, $i = 1, 2$, where (D) holds by the Duality Theorem 3.1. It then uses various measure theoretic properties as the outer measure property of Pachl, the Marczewski imbedding theorem, and a measure extension property.

In the following we deal with the problem whether perfectness is also a necessary condition. To study this question we introduce the notion of a strong duality space.

Definition 3.4 $(U_1, \mathfrak{A}_1, P_1)$ is a *'strong duality space' if it is a duality space and the functional I is stable under extensions, i.e. for any $(U_2, \mathfrak{A}_2, P_2)$ and any sub σ-algebra $\mathfrak{C}_2 \subset \mathfrak{A}_2$ and $h \in B(\mathfrak{A}_1 \otimes \mathfrak{C}_2)$ holds*

$$I_{\mathfrak{A}_1 \otimes \mathfrak{C}_2}(h) = I_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}(h). \tag{3.33}$$

Equivalently one could also postulate stability of S. As consequence of the general duality theorem one obtains that *perfectness* implies *strong duality space.* We need two further measure theoretic properties.

Definition 3.5 *$(U_1, \mathfrak{A}_1, P_1)$ has the 'projection property' if for all $(U_2, \mathfrak{A}_2, P_2)$ and $C \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$ there exists $A_1 \in \mathfrak{A}_1$ with $P_1(A_1) = 1$ and*

$$\pi_2(C \cap (A_1 \times U_2)) \in \overline{\mathfrak{A}_2}^{P_2}, \ \overline{\mathfrak{A}_2}^{P_2} \text{ the } P_2\text{-completion of } \mathfrak{A}_2. \tag{3.34}$$

This notion is a measure theoretic analog of the projection property in descriptive set theory. The classical result in this area says that the projection of a Borel set in a product of two standard Borel spaces is analytic and thus universally measurable.

The second property is the measure extension property.

Definition 3.6 *$(U_1, \mathfrak{A}_1, P_1)$ has the 'measure extension property' if for all $(U_2, \mathfrak{A}_2, P_2)$ for all $\mathcal{D}_2 \subset \mathfrak{A}_2$ and all $P \in M(P_1, P_2/\mathcal{D}_2)$ there exists an extension $\overline{P} \in M(P_1, P_2)$ such that $\overline{P}/\mathfrak{A}_1 \otimes \mathcal{D}_2 = P$.*

We say that $(U_1, \mathfrak{A}_1, P_1)$ has the 'charge extension property' if the extension can be found in the set $\operatorname{ba}(P_1, P_2)$ *of charges, i.e. non-negative finitely additive measures, with marginals P_i.*

It now turns out that the strong duality spaces are exactly the perfect spaces. Thus a general duality theorem in the strong sense implies perfectness. The following theorem states that the strong duality property is even equivalent with any of the measure theoretic notions introduced above.

Theorem 3.7 (Characterization theorem.) *(Ramachandran and Rü (2000)) For a probability space $(U_1, \mathfrak{A}_1, P_1)$ the following statements a)–e) are equivalent:*

a) $(U_1, \mathfrak{A}_1, P_1)$ is a strong duality space.

b) $(U_1, \mathfrak{A}_1, P_1)$ is perfect.

c) $(U_1, \mathfrak{A}_1, P_1)$ has the measure extension property.

d) $(U_1, \mathfrak{A}_1, P_1)$ has the projection property.

e) $(U_1, \mathfrak{A}_1, P_1)$ has the charge extension property.

As consequence all structure theorems for perfect spaces are also valid for strong duality spaces. On the other hand this result says that one cannot expect 'good' duality results on 'general' infinite dimensional spaces. There remain the following important

Open problems:

a) Is any measure space a duality space?

b) Is any duality space a strong duality space?

c) Is $M(P_1, P_2) \subset \operatorname{ba}(P_1, P_2)$ dense in weak $*$-topology from $B(U_1 \times U_2)$?

4 Optimal multivariate couplings

The duality results of section 3 were developed into more concrete optimal coupling results in the early nineties. Here also started the development on the subject by several researchers from analysis since the strong and fruitful connections to several problems in analysis soon became clear. In particular to mention is the work of Brenier, McCann, Gangbo and later on Ambrosio, Villani, Otto, Caffarelli, Evans, Trudinger, Lott, and Sturm.

4.1 The squared norm cost

For the squared norm cost $c(x,y) = \|x-y\|^2$, $x, y \in \mathbb{R}^k$ the problem of optimal transport or optimal couplings is given by

$$E\|X - Y\|^2 = \inf_{X \sim P_1, Y \sim P_2} \tag{4.1}$$

where $P_i \in M^1(\mathbb{R}^k, \mathfrak{B}^k)$ have covariance matrices $\sum_i = \mathrm{Cov}(P_i)$. This problem is equivalent to maximizing the trace tr Ψ

$$\mathrm{tr}\ \Psi = \max! \tag{4.2}$$

over all

$$\Psi \in C(P_1, P_2) = \left\{ \Psi : \exists P \in M(P_1, P_2) \text{ such that } \begin{pmatrix} \Sigma_1 & \Psi \\ \Psi^T & \Sigma_2 \end{pmatrix} \in \mathrm{Cov}(P) \right\}. \tag{4.3}$$

In general $C(P_1, P_2)$ is a complicated set but for normal distributions $P_i = N(a_i, \sum_i)$ one gets the maximal possible class $C(P_1, P_2)$ with covariance matrices $\sum_i$ of P_i. The covariance condition (4.3) is in this case equivalent to

$$\begin{pmatrix} \Sigma_1 & \Psi \\ \Psi^T & \Sigma_2 \end{pmatrix} \geq 0 \text{ in the sense of positive semidefiniteness.} \tag{4.4}$$

The corresponding optimization problem (4.2) was analytically solved in Olkin and Pukelsheim (1982) and Dowson and Landau (1982), leading in particular to an universal lower bound of $\ell_2(P_1, P_2)$ depending only on first and second moments a_i, Σ_i for any pair $P_1, P_2 \in M^1(\mathbb{R}^k, \mathfrak{B}^k)$.

For general distributions P_i the following is the basic optimal coupling result which is due to Knott and Smith (1984, 1987), Rü and Rachev (1990), and Brenier (1987, 1991).

Theorem 4.1 (Optimal L^2-couplings.) *Let $P_i \in M^1(\mathbb{R}^k, \mathfrak{B}^k)$ with $\int \|x\|^2 dP_i(x) < \infty$, then*

a) *There exists an optimal L^2-coupling, i.e. a solution of* (4.1).

b) *$X \stackrel{d}{=} P_1$, $Y \stackrel{d}{=} P_2$ is an optimal L^2-coupling*

$$\Longleftrightarrow \exists \text{ convex, lsc } f \in L^1(P_1) \text{ such that } Y \in \partial f(X) \text{ a.s.} \tag{4.5}$$

c) *If $P_1 \ll \lambda^k$, then for f as in b)*

$$\partial f(X) = \nabla f(X) \text{ a.s. and } (X, \nabla f(X)) \tag{4.6}$$

is a solution of the Monge problem.

d) *If $P_1 \ll \lambda^k$, then there exists a P_1 a.s. unique gradient ∇f of a convex function f, such that*

$$P_1^{\nabla f} = P_2. \tag{4.7}$$

Remarks 4.2 a) Part b) of this theorem was given in this form first in Rü and Rachev (1990). The proof was based on the duality theorem. The sufficiency part for b) is contained already in Knott and Smith (1984, 1987). Brenier (1991) established the uniqueness result in d) as well as b) while a special version of c) is already in his 1987 paper. Note that the existence of a Monge solution in c) is an immediate consequence of b) and a.s. differentiability of convex functions (see Rockafellar (1970)). By this history it seems appropriate to describe this important theorem to the authors from probability and analysis as mentioned above.

b) **Cyclically monotone support.** By convex analysis condition (4.5) is equivalent to *cyclically monotone support* Γ of the optimal transportation measure $\mu = P^{(X,Y)}$, i.e. $\forall (x_1, y_1), \dots, (x_m, y_m) \in \Gamma$ holds

$$\sum_{i=1}^{m} y_i x_{i+1} \leq \sum_{i=1}^{m} y_i x_i \tag{4.8}$$

with $x_{m+1} := x_1$. This equivalence lead Gangbo and McCann (1995) to a new strategy of proof. If uniqueness holds (as in the case $P_1 \ll \lambda^k$) then *cyclical monotonicity* of the support of μ implies optimality. In this way they were able to replace the moment assumptions on P_1, P_2 by the uniqueness condition.

c) For $P_1 = f\lambda^k$, $P_2 = g\lambda^k$ absolutely continuous w.r.t. Lebesgue measure Caffarelli (1992, 1996) established regularity estimates of the optimal Monge solution Φ:
If $f, g \in C^{k,\alpha}$ (i.e. the partial derivatives up to order k are of Hölder type α) and $g > 0$ then $\Phi \in C^{k+2,\alpha}$. In particular if $f, g \in C^\infty$ and locally bounded from below, their supports and supp g is convex, then $\Phi \in C^\infty$ and Φ is a classical solution of the Monge-Ampère equation

$$\det D^2(x) = \frac{f(x)}{g(\nabla(x))} \quad [P_1] \tag{4.9}$$

(see Villani (2006) for more details).

A corollary of the optimal L^2-coupling theorem is the polar factorization theorem due to Brenier (1987).

Corollary 4.3 (Polar factorization theorem.) *Let $E \subset \mathbb{R}^d$ be a bounded subset with positive Lebesgue measure, $h : E \to \mathbb{R}^d$ a measurable map with $P^h \ll \lambda^d$, where $P = \lambda_E$ is the normalized Lebesgue measure on E. Then there exists a unique gradient ∇f of a convex lsc function f and a measure preserving map s on (E, P) such that*

$$h = \nabla f \circ s \ [P]. \tag{4.10}$$

Remarks 4.4 a) The nondegeneracy condition of the polar factorization theorem has been weakened by Burton and Douglas (1998, 2003). Also a counter-example is given there to show that the theorem is false without any further assumption.

b) The nondegeneracy condition of h in the polar factorization theorem can also be replaced by the following independence assumption $(\mathrm{I_h})$:

$(\mathbf{I_h})$ There exists a random variable V on (E, P) such that V is independent of h and $P^V = U(0, 1)$ is the uniform distribution on $(0, 1)$.

For the proof let (X, Y) be an optimal coupling of (P^h, P), $X \overset{d}{=} P^h$, $Y \overset{d}{=} P$. Since $P \ll \lambda^d$, by the optimal coupling result Theorem 4.1 there exists a unique gradient ∇f of a convex function f such that

$$X = \nabla f \circ Y \ \text{a.s..} \tag{4.11}$$

Now one can apply the following result of Rü (1985) on the solutions of *stochastic equations* (see also Rachev and Rü (1990):

Let $(E, \mu) \xrightarrow{h} B$, B a Borel space, let $(F, R) \xrightarrow{f} B$, h, f measurable, μ, R probability measures, and let (E, μ) be rich enough (i.e. it allows a uniformly distributed r.v. V on (E, μ) independent of h). If the distributional equation

$$\mu^h = R^f \tag{4.12}$$

holds, then there exists an r.v. $U : E \to F$ with $\mu^U = R$ such that the stochastic equation

$$h = f \circ U \ \ [\mu] \tag{4.13}$$

holds.

Applying this general factorization theorem with $\mu = \lambda_E^d = P$, $R = P = \mathcal{L}(Y)$ we obtain the existence of a measurable factorization

$$h = \nabla f \circ U \ \ [P] \tag{4.14}$$

with some measure preserving map U on (E, P), i.e. the polar factorization result.

Corollary 4.5 *For any measurable function h the independence hypothesis* $(\mathrm{I_h})$ *implies the existence of a polar factorization.*

In general the independence hypothesis does not hold. If e.g. $d = 1$ and $h(u) = u$, $u \in [0, 1] = E$ then I_h does not hold. If $V = V(u)$ would be independent of h, then $P^{V|h=u} = \varepsilon_{V(u)}$, a contradiction. But by enlarging E to e.g. $E' := E \times [0, 1]$ and considering $P' = P \otimes \lambda\backslash_{[0,1]}$ we can consider h formally as function on E' by $h(x, u) := h(x)$. The independence hypothesis holds in this extended framework and thus there exists an r.v. U on E' such that $P'^U = P = \lambda_E$ and

$$h = \nabla f \circ U \ \ [P'], \tag{4.15}$$

i.e. $h(x) = \nabla f \circ U(x, u) \ \ [P']$.

Thus, we obtain a polar factorization theorem in the 'weak sense' without any nondegeneration condition on h.

Corollary 4.6 *The polar factorization theorem holds in the extended sense* (4.15) *without any further nondegeneration assumption.*

This extension also holds for McCann's (2001) version of the polar factorization theorem in Riemannian manifolds.

4.2 General coupling function

For the case of general coupling (resp. cost) functions $c = c(x, y)$ and probability measures P, Q we consider the corresponding optimal coupling (transport) problem

$$S(c) = \sup\left\{ \int c\,d\mu; \mu \in M(P, Q) \right\} \tag{4.16}$$

with dual problem

$$I(c) = \inf\left\{\int h_1 dP + \int h_2 dQ; c \le h_1 \oplus h_2, h_i \in L^1\right\} \tag{4.17}$$

The following notions from nonconvex optimization theory as discussed in Elster and Nehse (1974) and Dietrich (1988) are useful and were introduced in the context of the transportation problem in Rü (1991b). A proper function $f : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$ is called c-convex if it has a representation of the form

$$f(x) = \sup_y (c(x,y) + a(y)) \tag{4.18}$$

for some function a. The *c-conjugate* f^c of f is defined by

$$f^c(y) = \sup_x (c(x,y) - f(x)), \tag{4.19}$$

the sup being over the domain of f. Defining further the *double c-conjugate* f^{cc} by

$$f^{cc}(x) = \sup_y (c(x,y) - f^c(y)) \tag{4.20}$$

then f^c, f^{cc} are c-convex, f^{cc} is the largest c-convex function majorized by f and $f = f^{cc}$ if and only if f is c-convex. The pair f^c, f^{cc} is an admissible pair in the sense that

$$f^c(y) + f^{cc}(x) \ge c(x,y) \text{ for all } x, y. \tag{4.21}$$

Obviously this construction is similarly possible on a general pair U_1, U_2 of spaces replacing $\mathbb{R}^k$ and $c : U_1 \times U_2 \to \mathbb{R} \cup \{\infty\}$. The (double) c-conjugate functions are basic for the theory of inequalities as in (4.21). The generalized *c-subgradient* of a function f at a point x is defined by

$$\partial_c f(x) = \{y; f(z) - f(x) \ge c(z,y) - c(x,y) \ \forall z \in \operatorname{dom} f\} \tag{4.22}$$

further

$$\partial_c f = \{(x,y) \in U_1 \times U_2; y \in \partial_c f(x)\}. \tag{4.23}$$

Denoting by $\mathfrak{C} := \{\Psi_{y,a}; y \in U_2, a \in \mathbb{R}\}$ the class of all shifts and translates of c, $\Psi_{y,a}(x) := c(x,y) + a$, c-convexity of f is equivalent to a representation of the form $f(x) = \sup_{\Psi \in \mathfrak{C}'} \Psi(x)$ for some $\mathfrak{C}' \subset \mathfrak{C}$ and further (with $a := f(x) - c(x, y_0)$) $y_0 \in \partial_c f(x)$ if and only if $\exists\, a \in \mathbb{R}$ such that

$$\Psi_{y_0,a}(x) = f(x) \tag{4.24}$$

$$\Psi_{y_0,a}(x') \le f(x'), \quad \forall x' \in \operatorname{dom} f. \tag{4.25}$$

This geometric description of c-subgradients generalizes the corresponding description in the case of convex functions and is very useful and intuitive.

The following is an extension of the optimal coupling (transportation) result in Theorem 4.1 to general cost functions and general measure spaces U_i as in the basic Duality Theorem 4.1.

Theorem 4.7 (Optimal c-couplings.) *(Rü (1991b)) Let c be a lower majorized function (i.e. $c(x,y) \ge f_1(x) + f_2(y)$ for some $f_1 \in L^1(P)$, $f_2 \in L^1(Q)$) and assume that*

$I(c) < \infty$. Then a pair (X, Y) with $X \stackrel{d}{=} P$, $Y \stackrel{d}{=} Q$ is an optimal c-coupling between P and Q if and only if

$$(X, Y) \in \partial_c f \ \text{ a.s.} \tag{4.26}$$

for some c-convex function f, equivalently, $Y \in \partial_c f(X)$ a.s.

The characterization in (4.26) is equivalent to the condition that the support Γ of the joint distribution of X, Y is *c-cyclically monotone*, i.e. for all $(x_i, y_i) \in \Gamma$, $1 \le i \le n$, $x_{n+1} := x_1$ holds

$$\sum_{i=1}^{n} \big(c(x_{i+1}, y_i) - c(x_i, y_i)\big) \le 0. \tag{4.27}$$

This notion was introduced in Smith and Knott (1992), who reformulated Theorem 4.7 in terms of c-cyclical monotonicity. For the equivalence see also Rü (1995, 1996b) and Gangbo and McCann (1996). Note that for the corresponding inf problem (transportation problem) the inequality sign in (4.27) has to be chosen in the opposite direction.

Without the moment assumptions in Theorem 4.7 c-cyclically monotone support is in general not a sufficient condition for optimality (see Ambrosio and Pratelli (2003)). Gangbo and McCann (1996) have developed a characterization of c-optimality based on c-cyclically monotone support plus a uniqueness property. They also have studied some regularity properties of c-convex functions. The moment assumptions of the duality theorem have been weakened in Ambrosio and Pratelli (2003) and Schachermayer and Teichmann (2006) for lower semicontinuous cost functions c. The notion of 'strongly c-monotone' support is introduced and shown as a sufficient condition in their paper.

The characterization of solutions of the optimal coupling problem and the results on existence of solutions (X, Y) resp. f imply the following necessary condition for differentiable cost functions: If $(X, Y), f$ are solutions of the optimal c-coupling problem on $\mathbb{R}^k$ and if $P \ll \lambda^k$ and f is differentiable almost everywhere, then

$$\nabla f(X) = \nabla_x c(X, Y) \ \text{ a.s.} \tag{4.28}$$

(see Rü (1991b)).

In this direction Gangbo and McCann (1996) have shown that if $c = c(x - y)$ is locally Lipschitz, then c-convex functions are differentiable almost everywhere. For an extension see (Villani, 2006, p. 125). As consequence one obtains:
If (4.28) has a unique solution in $Y = \Phi(X)$, then

$$(X, \Phi(X)) \ \text{ is a Monge solution.} \tag{4.29}$$

In the case where $c(x, y) = h(x - y)$, h strictly convex, c-convex functions f are convex and thus ∇f exists almost everywhere and then (4.28) implies

$$Y = X - \nabla h^*(\nabla f(X)) =: \Phi(X) \tag{4.30}$$

where h^* is the convex conjugate of h (see Gangbo and McCann (1996)). A similar example for the concave functions $h(|x - y|)$ is in Rü and Uckelmann (2000). Note that Monge solutions for (4.16) are solutions in weak sense for generalized PDE's of Monge-Ampère type.

Remarks 4.8 a) Call a function Φ c-cyclically monotone if $\Gamma = \operatorname{graph}(\Phi)$ is c-cyclically monotone. Several sufficient and several necessary conditions for Φ to be c-cyclically monotone have been given in Smith and Knott (1992) and in Rü (1995). For the optimal ℓ_p-coupling with $c(x,y) = \|x-y\|^p$, $\|\cdot\|$ the Euclidean norm, one obtains e.g. for $p > 1$ that for h cyclically monotone

$$\Phi(x) := \|h(x)\|^{-\frac{p-2}{p-1}} h(x) + x \tag{4.31}$$

is c-cyclically monotone. Similar extensions hold for p-norms $\|\cdot\|_p$. In particular it is shown that radial transformations

$$\Phi(x) = \alpha(\|x\|)\frac{x}{\|x\|}, \quad \alpha(t) \geq t \tag{4.32}$$

are optimal. This allows e.g. to determine optimal couplings between uniform distributions on two ellipsoids. For the Euclidean norm $\|\cdot\| = \|\cdot\|_2$ and $p = 1$, the classical Monge case, one obtains that the optimal transport is concentrated on lines $Y \in \{X + t\nabla f(X); 0 \leq t \leq T\}$. It is however not uniquely determined by this property. Existence of Monge-solutions for the classical Monge case where c is the Euclidean norm has a long history, starting with early work of Sudakov (1979) (for details see Villani (2006)).

b) **One-dimensional case.** In the one-dimensional case the optimal coupling result in Theorem 4.7 has been applied to determine optimal couplings for some classes of nonconvex functions, see Uckelmann (1997), Rü and Uckelmann (2000), and, based on a direct analysis of monotonicity properties, in McCann (1999). In the case where $P = Q = U(0,1)$ and $c(x,y) = \Phi(x+y)$ and for coupling functions Φ such that

$$\Phi''(t)\begin{cases} > 0 & t \in [0,k_1) \cup (k_2, 2] \\ < 0 & t \in (k_1, k_2), \end{cases} \tag{4.33}$$

i.e. Φ is of the type: convex-concave-convex, the following result was proved in Uckelmann (1997):

Proposition 4.9 *If α, β are solutions of*

$$\begin{cases} \Phi(2\alpha) - \Phi(\alpha+\beta) + (\beta-\alpha)\Phi'(\alpha+\beta) = 0, \\ \Phi(2\beta) - \Phi(\alpha+\beta) + (\alpha-\beta)\Phi'(\alpha+\beta) = 0, \end{cases} \tag{4.34}$$

$0 < \alpha < \beta < 1$, *then $(X, T(X))$ is an optimal c-coupling where*

$$T(x) = \begin{cases} x, & x \in [0,\alpha] \cup [\beta, 1], \\ \alpha + \beta - x, & x \in (\alpha, \beta). \end{cases} \tag{4.35}$$

Similar results have been shown for $c(x,y) = \Phi(|x-y|)$, Φ convex-concave for the case of uniform marginals. Extensions of these results are in Rü and Uckelmann (2000). Also some results for nonuniform marginals and numerical results are given there. These results confirm some general conjectures on the solutions in the one dimensional case. The case $c(x,y) = \Phi(|x-y|)$, Φ concave was studied in McCann (1999) in detail.

c) **Discrete distributions, Voronoi cells.** In the case where one distribution is discrete $Q = \sum_{i=1}^n \alpha_j \varepsilon_{x_j}$ the relevant c-convex functions for the optimal couplings are of the form

$$f(x) = \max(c(x, x_j) + a_j). \tag{4.36}$$

The subgradients are to be the support points x_j of Q and we have to determine only the shifts a_j. Let

$$\begin{aligned} A_j &:= \{x : f(x) = c(x, x_j) + a_j\} \\ &= \{x : x_j \in \partial_c f(x)\} \end{aligned} \tag{4.37}$$

denote the corresponding Voronoi cells, then the subgradients are unique except at the boundaries of A_j. This observation implies a uniqueness result for discrete distributions (see Cuesta-Albertos and Tuero-Diaz (1993)). The optimal transportation problem between P and Q reduces to finding shifts a_j such that $P(A_j) = \alpha_j$, $1 \leq j \leq n$. For the case of $c(x,y) = \|x-y\|^a$ one gets for $a = 2$ linear boundaries. Some cases for $a = 1, 2, 4$ are dealt with explicitly in Rü and Uckelmann (1997, 2000) in the case where P is uniform on a square in $\mathbb{R}^2$ or a cube in $\mathbb{R}^3$. For the solution in not too large discrete cases one can apply sophisticated linear programming techniques or algorithms developed for Voronoi cells (see Rü and Uckelmann (2000)). For an alternative continuous time algorithm see Brenier (1999).

4.3 The n-coupling problem

The coupling problem between two distributions on $\mathbb{R}^k$ is naturally extended to the optimal coupling (transportation) between n probability measures $P_1, \dots, P_n$ on $\mathbb{R}^k$. This might be used as discrete time approximation of a continuous time flow along some time interval $[0, T]$ where P_i are intermediate distributions on the transport from P_1 to P_n. A new problem arises only if in the formulation of the coupling problem there is a cycle or a back coupling e.g. between P_n and P_1. For the L^2-cost one version of this problem is

$$E\Big\|\sum_{i=1}^n X_i\Big\|^2 = \max \tag{4.38}$$

over all $X_i \stackrel{d}{=} P_i$, $1 \leq i \leq n$.

For the case $n = 3$ and $P_i = N(0, \Sigma_i)$ this problem was posed in Olkin and Rachev (1993). It is for $n = 3$ equivalent to

$$E(\langle X, Y\rangle + \langle Y, Z\rangle + \langle X, Z\rangle) = \max! \tag{4.39}$$

and thus includes cycles: some mass should be cycled around three places from 1) to 2) from 2) to 3) but then also back from 3) to 1) in an optimal way. Knott and Smith (1994) (in the case $n = 3$) proposed the fruitful idea that "optimal coupling to the sum $T = \sum_{i=1}^n X_i$ should imply multivariate optimal coupling in the sense of (4.38)".

The reason for this conjecture is the equivalence of (4.38) with the problem

$$E\sum_{i=1}^{n} \|X_i - T\|^2 = \min_{X_i \overset{d}{=} P_i} ! \tag{4.40}$$

In the case of multivariate normal distributions $P_i = N(0, \Sigma_i)$, $1 \leq i \leq 3$, this idea leads then to the following algorithm: Let $T \overset{d}{=} N(0, \Sigma_0)$, Σ_0 positive definite, and define

$$X = S_1 T, \quad Y = S_2 T, \quad Z = S_3 T \tag{4.41}$$

the optimal 2-couplings with

$$S_i = \Sigma_i^{1/2} \big(\Sigma_i^{1/2} \Sigma_0 \Sigma_i^{1/2}\big)^{-1/2} \Sigma_i^{1/2}.$$

Then one needs that $T = X + Y + Z = (S_1 + S_2 + S_2)T$ and thus one needs the identity $S_1 + S_2 + S_2 = I$. This leads to the following nonlinear equation for Σ_0:

$$\sum_{i=1}^{3} \big(\Sigma_0^{1/2} \Sigma_i \Sigma_0^{1/2}\big)^{1/2} = \Sigma_0. \tag{4.42}$$

As consequence of this idea one gets the following result:

If (4.42) has a positive definite solution Σ_0, then X, Y, Z as defined in (4.41) are optimal solutions of the 3-coupling problem (see Knott and Smith (1994)).

The natural iterative algorithm for the solution of (4.42) converges rapidly in $d = 2$ (as reported in Knott and Smith (1994)) but in $d = 3$ it turns out that convergence depends very sensitive on the initial conditions.

It was shown only in Rü and Uckelmann (2002) based on the uniqueness result for optimal couplings that this idea of Knott and Smith is valid in the normal case (in particular (4.42) has a solution) for any $n \geq 3$.

For general distributions P_i however optimal coupling to the sum is not sufficient for optimal n-coupling. There are some simple counter-examples. But a positive result is given in Rü and Uckelmann (2002) saying that optimal coupling to the sum T, continuity of $P^T \ll \lambda^d$ and 'maximality' of the domain of P^T imply optimal n-coupling.

The paper contains also a simple proof of the existence result of Gangbo and Święch (1998) on Monge solutions (in the case without cycles) of the form

$$X = (X_1, \Phi_2(X_1), \ldots, \Phi_n(X_1)) \tag{4.43}$$

if $P_i \ll \lambda^d$ and gives a one-to-one equivalence of the n-coupling problem with several modified 2-coupling problems.

5 A variant of the minimal ℓ_s-metric – application to the analysis of algorithms

Many algorithms of recursive structure (divide and conquer type algorithms) allow to derive limit theorems for their important parameters by the *contraction method* (see

Rösler and Rü (2001) and Neininger and Rü (2004)). The limits then are characterized by stochastic fix point equations typically of the form

$$X \stackrel{d}{=} \sum_{i=1}^{K}(A_i X_i + b_i) \tag{5.1}$$

where X_i are independent copies of X and where (X_i) and the random coefficients (A_i, b_i) are also independent.

For example: The internal path length Y_n of the Quicksort algorithm satisfies the recursive equation

$$Y_n \stackrel{d}{=} I_n Y_{I_n} + (n - I_n - 1)\overline{Y}_{n-I_n-1} + (n-1), \tag{5.2}$$

where I_n is uniform on $\{0, \dots, n-1\}$ distributed and $(\overline{Y}_n)$ are independent copies of (Y_k).

The normalization $X_n := \frac{Y_n - EY_n}{n}$ satisfies the recursive equation

$$Y_n \stackrel{d}{=} \frac{I_n}{n} X_{I_n} + \frac{n - I_n + 1}{n}\overline{X}_{n-I_n+1} + C_n(I_n) \tag{5.3}$$

and leads to the limit equation

$$X \stackrel{d}{=} UX + (1-U)\overline{X} + C(U), \tag{5.4}$$

where the entropy function $C(x) = x \ln x + (1-x)\ln(1-x) + 1$ arises as the limit of C_n and $U \stackrel{d}{=} U[0,1]$ as the limit of $\frac{I_n}{n}$. The contraction method then gives general conditions which imply existence and uniqueness of solutions of (5.4) and convergence of X_n to this solution.

The solution of (5.4) resp. (5.1) can be described as fixpoint of the operator

$$T : M_s \to M_s, \quad T\mu = \mathcal{L}\Big(\sum_{i=1}^{n}(A_i X_i + b_i)\Big) \tag{5.5}$$

where (X_i) are iid and $X_i \stackrel{d}{=} \mu$. M_s is the class of distributions with finite s-th moments. A natural contraction condition is:

$$A_i, b_i \in L^s, \quad \sum_{i=1}^{K} E|A_i|^s < 1. \tag{5.6}$$

This condition has been shown to imply the fixpoint result for $0 \leq s \leq 1$ and for $s = 2$ (with the additional restriction of a fixed first moment). For $1 < s < 2$ one has only an inequality of the form

$$\ell_s^s(T\mu, T\nu) \leq K_s^s\Big(\sum_{i=1}^{K} E|A_i|^s\Big)\ell_s^s(\mu, \nu) \tag{5.7}$$

with some constant $K_s > 1$. This inequality yields only an existence result for the fixpoint equation (5.1) under restrictive conditions. To solve this problem a new modification of the minimal ℓ_s-metric was introduced in Rü (2006). Define

$$\ell_s^0(\mu, \nu) := \inf\{\|X - Y\|_s : X \stackrel{d}{=} \mu, Y \stackrel{d}{=} \nu, \text{ and } X \approx Y\} \tag{5.8}$$

Here $X \approx Y$ is defined by the conditions

$$E(X - Y) = 0, \quad E|X - Y|^s < \infty. \tag{5.9}$$

Note that condition (5.9) does not need that $\mu, \nu \in M_s$.

With this new variant of the minimal ℓ_s-metric which is defined by a modification of the transportation problem one obtains the existence of fixpoints under the natural condition (5.6) on the coefficients (see Rü (2006)).

Theorem 5.1 *If A_i, $b_i \in L^s$, $1 \leq s \leq 2$, $E\sum_{i=1}^K |A_i|^s < 1$ and if for some $\mu_0 \in M$, $\ell_s^0(\mu_0, T\mu_0) < \infty$, then there exists a unique solution of* (5.1) *in*

$$M_s^0(\mu_0) = \{\mu \in M;\ \ell_s^0(\mu, \mu_0) < \infty\}. \tag{5.10}$$

In the proof of this theorem it is shown that $(M_s^0(\mu_0), \ell_s^0)$ is a complete metric space and with the help of weighted branching processes it is shown that some power T^{n_0} of T satisfies a contraction condition on $M_s^0(\mu_0)$ w.r.t. the new variant ℓ_s^0 of the minimal ℓ_s-metric.

An interesting corollary of this result is an equivalence principle for homogeneous and inhomogeneous stochastic equations.

Corollary 5.2 *Under the assumptions of Theorem 5.1 there is a one-to-one relationship between solutions of the homogeneous stochastic equation*

$$X \stackrel{d}{=} \sum_{i=1}^{K} A_i X_i \tag{5.11}$$

and the inhomogeneous stochastic equation

$$Y \stackrel{d}{=} \sum_{i=1}^{K} (A_i Y_i + b_i). \tag{5.12}$$

More exactly: For any solution X of (5.11) *there exists exactly one solution Y of* (5.12) *with distribution $\mu \in M_s^0(\mu_0)$, where $\mu_0 = \mathcal{L}(X)$ and conversely.*

Thus the modification of the minimal ℓ_s-metric allows to investigate solutions of fixpoint equations (5.1) without any moment conditions.

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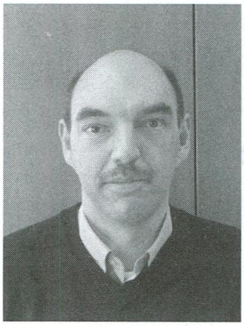

The Mathematics of Light Pulses in Dispersive Media

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Abstract

- Mathematics Subject Classification: 35 L 05, 35 L 15, 35 Q 60, 35 Q 55
- Keywords and Phrases: dynamics of wave packets, nonlinear optics, pulse interaction

In this survey article we discuss a number of mathematical questions related to the behavior of light pulses in dispersive media. Mathematically, we analyze the dynamics of modulating pulse solutions of a nonlinear wave equation. Such solutions consist of a pulse-like envelope advancing in the laboratory frame and modulating an underlying wave-train. We explain the role of the Nonlinear Schrödinger equation in the description of pulses with the same carrier wave. We show that there is almost no interaction of well prepared pulses with different carrier waves. Finally, we discuss the question: Do modulating pulse solutions exist for all times? We discuss the relevance of the presented results for fiber optics and photonic crystals.

Eingegangen: 24. 01. 2007

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1 Introduction

The transport of information over long distances through optical fibers is one of the key technologies of the post-industrial society. Information is encoded digitally by ones and zeroes, i.e., by sending a light pulse through the optical fiber or not. Physically such a light pulse is a complicated structure. It consists of an underlying electromagnetic carrier wave moving with phase velocity c_p and of a pulse-like envelope moving with group velocity c_g and modulating the underlying carrier wave, see Fig. 1.

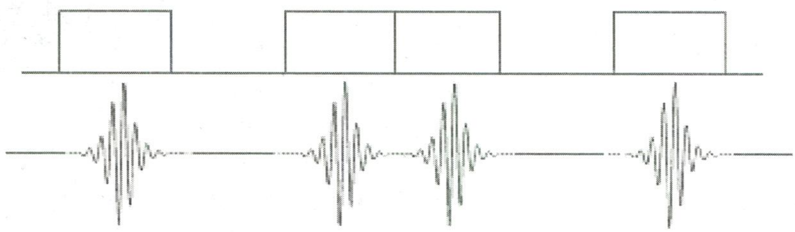

Figure 1: 0s and 1s are encoded physically by sending a light pulse or not; thus, for instance, the above electromagnetic wave encodes the sequence 101101.

The analysis of the evolution of such a light pulse is a nontrivial task. The system shows dispersion and (weak) dissipation, i.e., harmonic waves with different wave numbers travel at different speeds and energy is lost in a wave number dependent way. Moreover, there is a nonlinear response by the optical fiber. Thus, at a first glance it looks like a typical example for the application of numerical methods. However, a direct simulation of Maxwell's equations which describe these electromagnetic waves is beyond any present possibilities. This can be seen as follows: The wavelength of the underlying carrier wave is around 10^{-7} m. Resolving this structure in a fiber of 10 km $=10^4$ m gives in uniform one dimensional spatial discretization 10^{11} points, not to speak about the transverse directions and the temporal discretization. Therefore, before making any numerical investigations, the system has to be analyzed and simpler, numerically more suitable, models have to be derived. Interestingly, by using only a pencil and a sheet of paper a lot of things can be concluded without using any computer. This will be the subject of this survey article.

Using multiple scaling analysis we derive a formula for the optimal shape of the envelope of the pulse. Optimal means that it is more or less of a permanent form, i.e., in the ideal case the pulse is time periodic in a frame moving with the group velocity of the envelope. We will explain that the dynamics of pulses with the same carrier wave, i.e. with the same wave length, can be described by the dynamics of the envelope alone which is governed by a Nonlinear Schrödinger equation (NLS equation). The NLS equation is a universal nonlinear partial differential equation. Universal here means that additional to nonlinear optics it appears in the above sense in many contexts, for instance water waves, plasma physics, and lattice vibrations. Moreover, the NLS equation is a completely integrable Hamiltonian system. As a result, the NLS equation can (in principle) be solved explicitly. The method is called the inverse scattering scheme. In particular, the NLS equation has explicit so-called N-soliton solutions. These are special localized waves with N humps, $N \in \mathbb{N}$, where the humps interact asymptotically in a very unexpected way which is similar to the superposition principle in linear equations.

We will also explain that pulses with different carrier waves, i.e. different wave lengths, do not interact in lowest order. This fact allows to increase the information rate through the fiber by using different bands, i.e. a number of different carrier waves.

Photonic crystals play an important role in nanotechnological devices. One of the ultimate goals is to use them as optical storage. We will explain the possibility of standing light pulses in photonic crystals.

Finally, we will explain that the formula for the pulses of permanent form is correct to any polynomial order in the amplitude parameter, but that exponentially small terms will hinder the existence of a modulating pulse of permanent form with finite energy. However, it turns out that such modulating pulses of permanent form exist with infinite energy and exponentially small tails.

The paper starts with a short description of the physical background in order to motivate the description of nonlinear optics by nonlinear wave equations. We concentrate on rigorous mathematical results and skip in our presentation almost all purely formal results. We use ideas from finite and infinite dimensional dynamical systems theory, from perturbation theory and from a functional analytic treatment of partial differential equations over unbounded domains in Sobolev spaces.

The subsequent methods and results are not restricted to models from nonlinear optics. They essentially apply to all equations for which a NLS equation can be derived as an amplitude equation. For systems with (significant) dissipation the role of the NLS equation is taken by other but related amplitude equations, for instance of Ginzburg-Landau type. We refrain from any details in case of dissipation and refer to the literature, for instance [Sch99, Mie02] and the references therein.

Acknowledgment. The work is partially supported by the Deutsche Forschungsgemeinschaft DFG and the Land Baden-Württemberg through the Graduiertenkolleg GRK 1294/1: Analysis, Simulation und Design nanotechnologischer Prozesse.

2 Physical background

Light pulses are electromagnetic waves and described by Maxwell's equations, namely

$$\nabla \cdot \vec{B} = 0 \quad , \quad \nabla \times \vec{E} + \partial_t \vec{B} = 0,$$
$$\nabla \cdot \vec{D} = \rho \quad , \quad \nabla \times \vec{H} - \partial_t \vec{D} = \vec{J},$$

with $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ and $\vec{H} = \vec{B}/\mu_0 - \vec{M}$. Here $\vec{E} = \vec{E}(\vec{x}, t)$ is the electric field, $\vec{x} = (x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$ is the time, ε_0 the permittivity of vacuum, $\vec{P}$ the material polarization, $\vec{B}$ the magnetic flux, μ_0 the magnetic permeability of vacuum, $\vec{M}$ the material magnetization, ρ the charge density and $\vec{J}$ the electric current. These equations have to be closed with constitutive laws $\vec{P} = \vec{P}(\vec{E}, \vec{H})$ and $\vec{M} = \vec{M}(\vec{E}, \vec{H})$ describing the behavior of the medium. Depending on this choice there are linear and nonlinear, instantaneous and history dependent, dispersive and dissipative models.

In typical optical fibers there is no magnetization $\vec{M}$, no charge density ρ, and no electric current $\vec{J}$, and therefore, using $\nabla \times \nabla \vec{E} = \Delta \vec{E} - \nabla(\nabla \cdot \vec{E})$, Maxwell's equations for light in nonlinear optical material are given by

$$\triangle \vec{E} - \nabla(\nabla \cdot \vec{E}) - \partial_t^2 \vec{E} = \partial_t^2 \vec{P}, \tag{1}$$

where we scaled the speed of light in vacuum and the dielectric constant to 1.

The constitutive law for the polarization $\vec{P} = \vec{P}_{\mathrm{l}} + \vec{P}_{\mathrm{nl}}$ splits into a linear and a nonlinear part, which in general both depend on the history of the electric field. In centrosymmetric isotropic bulk material, the constitutive law for the linear response $\vec{P}_{\mathrm{l}}$ is given by an instantaneous part $\vec{P}_{\mathrm{l}}^{\mathrm{i}}(\vec{x}, t) = \vec{P}_{\mathrm{l}}^{\mathrm{i}}(\vec{x}, \vec{E}(\vec{x}, t))$ and a history dependent term

$$\vec{P}_{\mathrm{l}}^{\mathrm{h}}(\vec{x}, t) = (\chi_1 *_t \vec{E})(\vec{x}, t) = \int_{-\infty}^{\infty} \chi_1(t - \tau)\vec{E}(\vec{x}, \tau)\, \mathrm{d}\tau, \tag{2}$$

where χ_1 in (2) is a scalar function, independent of $\vec{x}$, with $\chi_1(t) = 0$ for $t < 0$ due to causality, and similar for the nonlinear polarization. In case of optical fibers χ_1 does also depend on the transverse directions y, z, and in case of photonic crystals also on the longitudinal direction x.

In the simplest case $\vec{E}$ is linearly polarized and only depends on x, i.e.,

$$\vec{E}(\vec{x}, t) = u(x, t)\hat{k} \quad \text{with} \quad \|\hat{k}\|_{\mathbb{R}^3} = 1, \quad (1, 0, 0) \cdot \hat{k} = 0. \tag{3}$$

Then, (1) simplifies to

$$\partial_t^2 u(x, t) = \partial_x^2 u(x, t) - \partial_t^2 p_{\mathrm{l}}(x, t) - \partial_t^2 p_{\mathrm{nl}}(x, t), \tag{4}$$

with $u(x, t), p_{\mathrm{l}}(x, t), p_{\mathrm{nl}}(x, t) \in \mathbb{R}$ such that $\vec{P}_{\mathrm{l}}(t, \vec{x}) = p_{\mathrm{l}}(x, t)\hat{k}$, $\vec{P}_{\mathrm{nl}}(t, \vec{x}) = p_{\mathrm{nl}}(x, t)\hat{k}$. The symmetry $(y, z) \mapsto -(y, z)$, which is present in most optical materials, prevents the occurrence of even terms in p with respect to u, thus, in general p_{nl} starts with cubic terms.

Due to the fact that we are mainly interested in the underlying mathematical structures, throughout the rest of the paper we choose

$$\partial_t^2 p(x, t) = u(x, t) - u^3(x, t)$$

as constitutive law, thus the toy problem for this paper is

$$\partial_t^2 u = \partial_x^2 u - u + u^3. \tag{5}$$

This choice is rather unphysical; however, it delivers a system with all properties in which we are interested, namely dispersive and nonlinear behavior. We refer to [SU03] for a mathematical discussion of a physically more realistic choice which includes dissipation and history dependence additional to dispersion and nonlinearity. Dissipation, i.e., wave number dependent damping, is usually very weak in the so-called transmission windows of optical fibers. However, it may become important over very long scales, while history dependence does not alter the analysis in an essential way.

3 Single pulses I

The description of light pulses, i.e. here of localized solutions of (5), is based on the derivation of a NLS equation by formal perturbation analysis. Therefore we introduce a small perturbation parameter

$$0 < \varepsilon \ll 1$$

which will relate the amplitude with the spatial and temporal scales. We seek $\mathcal{O}(\varepsilon)$-amplitude solutions which are slow spatial and temporal modulations of an underlying wave train $e^{i(k_0 x - \omega_0 t)}$, where k_0 and ω_0 are related by the dispersion relation $\omega_0^2 = k_0^2 + 1$ of the linearized problem $\partial_t^2 u = \partial_x^2 u - u$. Thus we substitute the ansatz

$$u_A(x,t) = \varepsilon(A(X,T)e^{i(k_0 x - \omega_0 t)} + \text{c.c.}) + \mathcal{O}(\varepsilon^2), \tag{6}$$

into (5), where $X = \varepsilon(x - ct)$ with c to be determined, where $T = \varepsilon^2 t$, where c.c. means complex conjugate, and where $A(X,T)$ is a complex-valued amplitude. We sort the coefficients of the carrier wave $e^{i(k_0 x - \omega_0 t)}$ with respect to powers of ε and obtain

$$\begin{array}{ll} \mathcal{O}(\varepsilon^1): & -\omega_0^2 A = -(k_0^2+1)A, \quad \text{dispersion relation}, \\ \mathcal{O}(\varepsilon^2): & 2ic\omega_0 A_X = 2ik_0 A_X \Rightarrow c = k_0/\omega_0 = \omega'(k_0) =: c_g', \quad \text{linear group velocity}, \end{array}$$

while at $\mathcal{O}(\varepsilon^3 e^{i(k_0 x - \omega_0 t)})$ we find that A should satisfy the NLS equation

$$2i\omega_0 \partial_T A + (1 - (c_g')^2)\partial_X^2 A + 3|A|^2 A = 0. \tag{7}$$

In fact, the Fourier transform of the ansatz (6) is strongly localized around k_0. Therefore, only the local shape of $\omega_{1,2} = \pm\sqrt{k^2+1}$ near k_0 is important to determine $c = c_g'$ in (6) and the coefficients of the linear terms in (7), see Fig. 2.

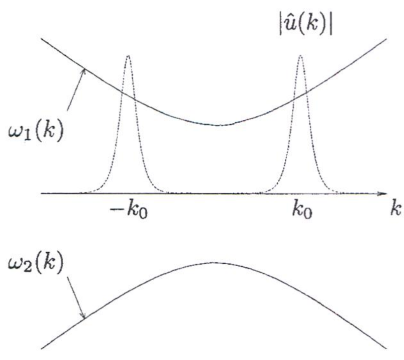

Figure 2: The two curves of eigenvalues $\omega_{1,2} = \pm i\sqrt{1+k^2}$. The Fourier transform of (6) is concentrated in an $\mathcal{O}(\epsilon)$ neighborhood around $\pm k_0$. Therefore, the dynamics of (6) is determined by the expansion of ω_1 at k_0. $\omega_1'(k_0)$ gives the linear group velocity c_g', and the group velocity dispersion $\omega_1''(k_0)$ occurs as coefficient in the NLS equation. The concentration of Fourier modes $\hat{u}(k)$ is respected by the nonlinear interaction, i.e. convolution in Fourier space.

Equation (7) has a four dimensional family of solutions of the form

$$A(X,T) = \tilde{A}(X - vT - X_0)e^{i(\tilde{v}X - \gamma_0 T + \phi_0)}, \quad \tilde{v} = (\omega_0 v)/(1-(c_g')^2),$$

in which the real-valued function $\tilde{A}$ satisfies the second-order ordinary differential equation

$$\partial_X^2 \tilde{A} = C_1 \tilde{A} - C_2 \tilde{A}^3, \tag{8}$$

where

$$C_1 = \tilde{v}^2 - \frac{2\gamma_0\omega_0}{1-(c_g')^2}, \quad C_2 = \frac{3}{(1-(c_g')^2)}.$$

Since $c_g' < 1$, we always have $C_2 > 0$, and for $C_1 > 0$ there exist the two explicit solutions

$$\tilde{A}_{\text{pulse}}(X) = \pm\left(\frac{2C_1}{C_2}\right)^{1/2} \operatorname{sech}\,(C_1^{1/2}X) \tag{9}$$

to (8). These are called homoclinic since they connect the origin $(0,0)$ as a fixed point of the first order formulation of (8) with itself, see the left panel of Fig. 3, while solutions which connect different fixed points of a dynamical system are called heteroclinic.

The derivation of the NLS equation (7) was only formal in the sense that we simply ignored terms that are higher order w.r.t. ε or appeared at a different wave-number. They are contained in the residual, i.e.

$$\text{Res}(u) := -\partial_t^2 u + \partial_x^2 u - u + u^3$$

contains the terms which do not cancel after inserting an approximation into (5). If $\text{Res}(u)=0$, then u is an exact solution of (5). It is important to note that due to a possible 'accumulation of errors' the smallness of the residual alone does not imply the so-called validity of the approximation where validity means that there are solutions of (5) which behave as predicted by the NLS equation on the relevant $\mathcal{O}(1/\varepsilon^2)$ time-scale.

However, there are a number of mathematical validity results for (5), see [Kal88, KSM92, Sch98] and also §4. The above procedure thus identifies modulating pulse solutions of (5) which are described by the approximate formula

$$\begin{aligned} u_{\text{pulse}}(x,t) &= \varepsilon\Big(\tilde{A}_{\text{pulse}}(X - vT - X_0)e^{i(\tilde{v}X-\gamma_0T+\phi_0)}e^{i(k_0x-\omega_0t)} + \text{c.c.}\Big) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon\Big(\tilde{A}_{\text{pulse}}(\varepsilon(x - c_g't - x_0 - \varepsilon vt))e^{i((k_0+\varepsilon\tilde{v})x-(\omega_0+\varepsilon^2\gamma_0)t+\phi_0)} + \text{c.c.}\Big) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

accurately over time-scales of order $\mathcal{O}(1/\varepsilon^2)$. In particular, for w.l.o.g. $v=0$, $x_0=0$ and $\phi_0 = 0$ we have

$$u_{\text{pulse}}(x,t) = \varepsilon\Big(\tilde{A}_{\text{pulse}}(\varepsilon(x-c_g't))e^{ik_0(x-(c_p'+\gamma_1\varepsilon^2)t)} + \text{c.c.}\Big) + \mathcal{O}(\varepsilon^2),$$

where $c_p' = (1+k_0^2)^{1/2}/k_0$ is the linear phase velocity and $\gamma_1 = \gamma_0/k_0$.

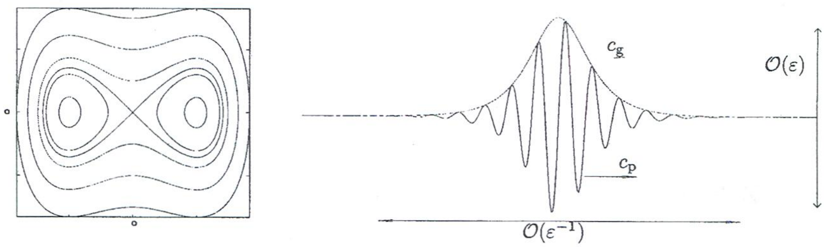

Figure 3: The $(\tilde{A}, \partial_X \tilde{A})$-phase portrait for (8), and a modulating pulse for (5) described by the NLS equation.

For the transport of information the global existence of modulating pulse solutions would be an important goal, i.e., we investigate if there are exact solutions to equation (5) of the form

$$u(x,t) = v(x - c_g t, k_0(x - c_p t)),$$

where v is 2π-periodic in its second argument with

$$\lim_{\xi \to \pm\infty} v(\xi, y) = 0.$$

This question will be discussed in detail in §7. As a first result we note [GS01] that such solutions can be computed approximately to any polynomial order in ε by extending the ansatz (6) by higher order terms and applying a small correction to the linear group speed, i.e., using $c_g = c_g' + \mathcal{O}(\varepsilon^2)$, see the right panel of Fig. 3. In other words, the following Lemma allows to find modulating pulse solutions which make the residual arbitrarily small.

To measure the residual we use Sobolev spaces [Ada75] $H^s = H^s(\mathbb{R}, \mathbb{C})$. For simplicity we restrict to $s \in \mathbb{N}$. Then H^s consists of all functions $f : \mathbb{R} \to \mathbb{C}$ which together with their distributional derivatives up to order s are square integrable, equipped with the norm

$$\|u\|_{H^s} = \sum_{j=0}^{s} \|\partial_x^j u\|_{L^2}.$$

In our spatially one-dimensional setting, H^s is a subset of the space of uniformly bounded and m times continously differentiable functions $C_b^m(\mathbb{R}, \mathbb{C})$ if $s > m + 1/2$, $m \in \mathbb{N}$, and the embedding is continuous, i.e., $\|u\|_{C_b^m} \leq C\|u\|_{H^s}$. These so-called Sobolev embeddings can be used to show that nonlinear terms such as u^3 are well-defined and continuous mappings from $H^s \to H^s$ if $s > 1/2$.

Lemma 3.1 *Let $s \geq 2$, $k_0 > 0$, $n \in \mathbb{N}$, and $\gamma_1 < 0$. For sufficiently small $\varepsilon > 0$ there exists a two-dimensional family, parameterized by envelope shift $x_0 \in \mathbb{R}$ and phase shift $\phi \in [0, 2\pi)$, of approximate modulating pulse solutions to (5) of the form*

$$u(x,t) = \varepsilon v_{k_0}(x - c_g t - x_0, k_0(x - c_p t) + \phi),$$

where v is 2π-periodic in its second argument, $c_{\mathrm{p}} = c'_{\mathrm{p}} + \gamma_1 \varepsilon^2$, $c_{\mathrm{g}} = 1/c_{\mathrm{p}}$, and where, for some $r > 0$, $\varepsilon v_{k_0}(\xi, y) = \varepsilon \tilde{A}_{\mathrm{pulse}}(\varepsilon\xi)\mathrm{e}^{\mathrm{i}y} + \mathcal{O}(\varepsilon^3 \mathrm{e}^{-r\varepsilon|\xi|}) + c.c.$, and

$$\|\mathrm{Res}(\varepsilon v_{k_0})\|_{H^s} \leq C\varepsilon^{n+1/2}.$$

4 Interaction of pulses with the same frequency

By the derivation of the NLS equation for the nonlinear wave equation (5) not only modulating pulse solutions of the nonlinear wave equation are identified. The complete dynamics known for the NLS equation can also be expected to be found approximately in the nonlinear wave equation.

We refer to the excellent textbooks [AS81, DJ89, SS99] about the various dynamics known for the NLS equation. For our purposes the fact is essential that the NLS equation is a completely integrable Hamiltonian system. Hamiltonian means that (7) can be written as $\partial_T A = J\delta H(A)$ where $J = -\mathrm{i}/(2\omega_0)$ is a skew symmetric operator and δ denotes the variational derivative of the Hamiltonian

$$H(A) = \int \left[\frac{1 - c_g'^2}{2} |\partial_X A|^2 - \frac{3}{4} |A|^4 \right] dX.$$

An immediate consequence is that the Hamiltonian $H(A)$ is conserved by the flow of (7), but in fact various further properties follow. Completely integrable means here that there are infinitely many independent conserved quantities for (7), and that there exists a transformation which is called inverse scattering scheme and which uses these conserved quantities to map (7) to a linear system which can (in principle) be solved explicitly. As a result, there are explicit though somewhat lengthy formulae (similar to (9)) for so-called N-soliton solutions of the NLS equation. In general, N-solitons are localized solutions which consist of N humps and which for $t \to \pm\infty$ asymptote to N solitons with different speeds. In particular, the individual humps interact in a very special way which is rather unexpected in a nonlinear equation: asymptotically for $t \to \pm\infty$ the interaction preserves the shapes and speeds of the individual humps, and only alters the relative positions. Thus, the humps are similar to elastic particles, and this motivates the name soliton. The change of position after interaction is $\mathcal{O}(1)$ in the NLS equation and is called a pulse shift. Formally, the N-solitons yield modulating N-pulse solutions for the nonlinear wave equation (5) with $\mathcal{O}(1/\varepsilon)$ pulse-shifts after interaction, see Fig. 4.

However, as already said in §3, the formal derivation of the NLS equation for the nonlinear wave equation (5) alone does not imply that the dynamics found in the NLS equation can also be found in the nonlinear wave equation (5): There are amplitude equations derived in a formally correct way by multiscale analysis which do not reflect the dynamics of the original system, see, e.g., [Sch95]. We now discuss the validity of the approximation, that is, how well solutions of the nonlinear wave equation (5) can be approximated via the solutions of the NLS equation.

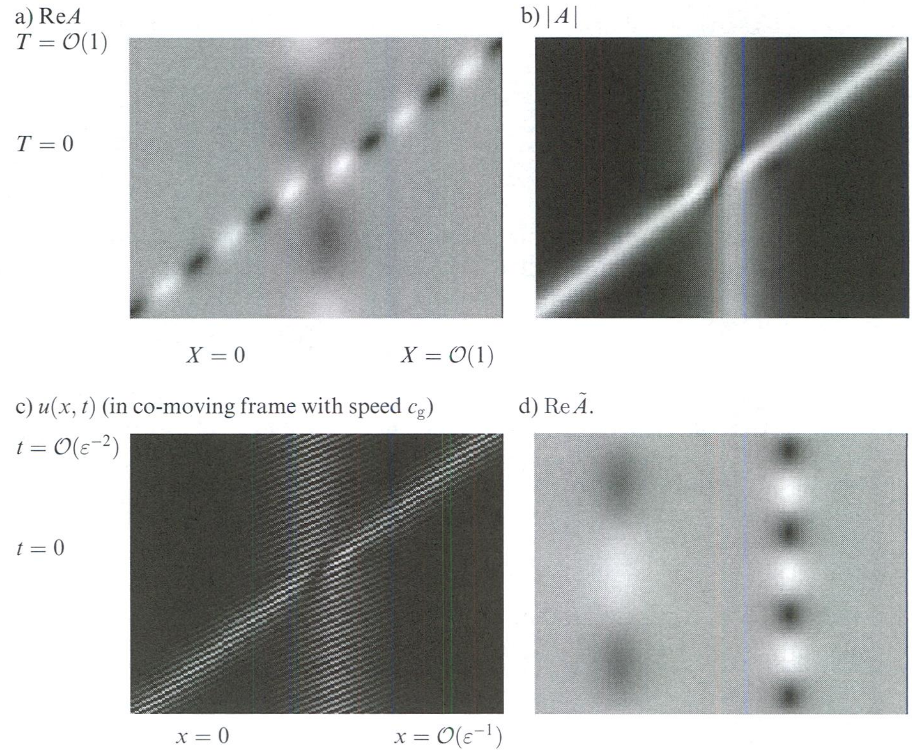

Figure 4: A 2-soliton A with interaction in the NLS equation and the associated modulating 2-pulse solution in the nonlinear wave equation (a)–(c), with interaction at $(X, T) = (0, 0)$, and a time periodic 2-solution $\tilde{A}$ in the NLS equation (d). For graphical reasons, black has been assigned to $u = 0$ in c).

Let $A \in C([0, T_0], H^{s_A}(\mathbb{R}, \mathbb{C}))$ be a solution of the NLS equation (7) with $s_A \geq 1$ defined below. Then

$$\varepsilon\psi_c(x, t) = \varepsilon A(\varepsilon(x - c_g' t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} + \text{c.c.} \tag{10}$$

defines a formal approximation of the solutions u of the nonlinear wave equation (5). For our purposes it turns out to be advantageous to consider the extended approximation

$$\varepsilon\psi(x, t) = \varepsilon A(\varepsilon(x - c_g' t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} + \varepsilon^3 A_3(\varepsilon(x - c_g' t), \varepsilon^2 t) e^{3i(k_0 x - \omega_0 t)} + \text{c.c.} \tag{11}$$

where $A_3 = A^3/(9\omega_0^2 - 9k_0^2 - 1)$ is also in $C([0, T_0], H^{s_A}(\mathbb{R}, \mathbb{C}))$, if $s_A \geq 1$. In summary, if $s_A > m + 1/2$, then there exist $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned}
\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi(\cdot,t)-\varepsilon\psi_c(\cdot,t)\|_{C_b^m} &\leq C \sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon^3 A_3(\varepsilon\cdot,\varepsilon^2 t)e^{3i(k_0x-\omega_0 t)}+c.c.\|_{C_b^m} \\
&\leq C\varepsilon^3 \sup_{t\in[0,T_0/\varepsilon^2]} \|A_3(\varepsilon\cdot,\varepsilon^2 t)\|_{C_b^m} \leq C\varepsilon^3 \sup_{T\in[0,T_0]} \|A_3(\cdot,T)\|_{C_b^m} \\
&\leq C\varepsilon^3 \sup_{T\in[0,T_0]} \|A_3(\cdot,T)\|_{H^{s_A}}
\end{aligned}$$

due to Sobolev's embedding theorem. As a consequence, if u can be approximated by $\varepsilon\psi$ up to an error of order $\mathcal{O}(\varepsilon^\beta)$ then it can also be approximated up to an error of order $\mathcal{O}(\varepsilon^{\min(3,\beta)})$ by ψ_c. In detail this means that

$$\|u-\varepsilon\psi_c\|_{C_b^m} \leq \|u-\varepsilon\psi\|_{C_b^m} + \|\varepsilon\psi-\varepsilon\psi_c\|_{C_b^m} \leq C\varepsilon^\beta + C\varepsilon^3 \leq 2C\varepsilon^{\min(3,\beta)}.$$

In order to estimate the difference $u-\varepsilon\psi =: \varepsilon^{3/2}R$ we derive an equation for R and estimate R. In order to do so we need estimates for the residual $\mathrm{Res}(\varepsilon\psi)$. By the choice of A and A_3 all terms up to formal order $\mathcal{O}(\varepsilon^4)$ are eliminated in the residual. Therefore there exist $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}(\varepsilon\psi(t))\|_{H^s} \leq C\varepsilon^{7/2}.$$

The loss of $\varepsilon^{1/2}$ comes from the scaling properties of the L^2-norm.

With $u = \varepsilon\psi + \varepsilon^{3/2}R$ we find

$$\partial_t^2(\varepsilon\psi+\varepsilon^{3/2}R) \quad = \quad \partial_x^2(\varepsilon\psi+\varepsilon^{3/2}R) - (\varepsilon\psi+\varepsilon^{3/2}R) - (\varepsilon\psi+\varepsilon^{3/2}R)^3$$

such that R satisfies

$$\partial_t^2 R = \partial_x^2 R - R + f \tag{12}$$

with

$$f = -3\varepsilon^2\psi^2 R - 3\varepsilon^{5/2}\psi R^2 - \varepsilon^3 R^3 + \varepsilon^{-3/2}\mathrm{Res}(\varepsilon\psi).$$

Thus,

$$\|f\|_{H^s} \leq C_1\varepsilon^2\|R\|_{H^s} + C_2(C_R)\varepsilon^{5/2}\|R\|_{H^s}^2 + C_3\varepsilon^2 \tag{13}$$

as long as $\|R(t)\|_{H^s} \leq C_R$ with a constant C_R determined below, constants C_1, C_3 independent of C_R and $\varepsilon \in (0,1)$ and a constant C_2 depending on C_R but independent of $\varepsilon \in (0,1)$.

The equation for R is solved here for simplicity with zero initial conditions. We use energy estimates and define the energy

$$E(R) = \sum_{j=0}^{s} \int_{-\infty}^{\infty} (\partial_t\partial_x^j R)^2 + (\partial_x^{j+1}R)^2 + (\partial_x^j R)^2 dx.$$

For $j = 0$ and $\int = \int_{-\infty}^{\infty}$ we obtain

$$\frac{1}{2}\partial_t \int (\partial_t R)^2 + (\partial_x R)^2 + R^2 dx = \int (\partial_t R)(\partial_t^2 R) + (\partial_x R)(\partial_t \partial_x R) + R(\partial_t R) dx$$
$$= \int [(\partial_t R)(\partial_x^2 R) - (\partial_t R)R + (\partial_t R)f + (\partial_x R)(\partial_t \partial_x R) + R(\partial_t R)] dx$$
$$= \int (\partial_t R) f dx$$

which can be estimated with the Cauchy Schwarz inequality by

$$\left|\int (\partial_t R) f \, dx\right| \leq \|\partial_t R\|_{L^2} \|f\|_{L^2} \leq \|\partial_t R\|_{L^2} (C_1 \varepsilon^2 \|R\|_{H^s} + C_2(C_R)\varepsilon^{5/2} \|R\|_{H^s}^2 + C_3 \varepsilon^2)$$
$$\leq C_1 \varepsilon^2 E(R) + C_2(C_R)\varepsilon^{5/2} E(R)^{3/2} + C_3 \varepsilon^2 E(R)^{1/2}$$
$$\leq (C_1 + C_3)\varepsilon^2 E(R) + C_2(C_R)\varepsilon^{5/2} E(R)^{3/2} + C_3 \varepsilon^2.$$

Since exactly the same estimates hold for $j = 1, \ldots, s$ we finally find

$$\partial_t E(R) \leq (C_1 + C_3)\varepsilon^2 E(R) + C_2(C_R)\varepsilon^{5/2} E(R)^{3/2} + C_3 \varepsilon^2. \tag{14}$$

Now assume that $\varepsilon^{1/2} C_2(C_R) E^{1/2}(R) \leq 1$. Then, for $0 \leq t \leq T_0/\varepsilon^{-2}$,

$$E(R(t)) \leq C_3 \mathrm{e}^{(C_1 + C_3 + 1)T_0} =: C_R^2 \tag{15}$$

by Gronwall's lemma which translates differential inequalities like (14) into pointwise estimates like (15), see, e.g., [Hen81, Lemma 7.1.1] for a very general version.

Choosing $\varepsilon_0 > 0$ so small that

$$\varepsilon_0^{1/2} C_2(C_R) C_R^{1/2} \leq 1 \tag{16}$$

we are done. In detail, to a given $C_R = C_R(T_0, C_1, C_3)$ defined in (15) we have a $C_2(C_R)$ by (13) and to this C_2 we have an $\varepsilon_0 > 0$ by (16). Hence, there are solutions u of (5) which behave for all $t \in [0, T_0/\varepsilon^2]$ as predicted by the NLS equation (7).

Theorem 4.1 *Fix $s_A \geq s + 3 \geq 4$. Let $A \in C([0, T_0], H^{s_A})$ be a solution of the NLS equation (7). There exist C, $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist solutions u of (5) such that $\sup_{t \in [0, T_0/\varepsilon^2]} \|u(\cdot, t) - \varepsilon\psi(\cdot, t)\|_{H^s} \leq C\varepsilon^{3/2}$.*

Remark 4.2 The time scale $\mathcal{O}(T_0/\varepsilon^2)$ is necessary to describe non-trivial dynamics. The error of order $\mathcal{O}(\varepsilon^{3/2})$ is much smaller than the approximation which is of order $\mathcal{O}(\varepsilon)$. Adding higher order terms, like A_3, to the approximation $\varepsilon\psi$ allows to decrease the magnitude of the residual further, in particular we can obtain $\mathcal{O}(\varepsilon^{11/2})$. This results in an error of order $\mathcal{O}(\varepsilon^{7/2})$ instead of $\mathcal{O}(\varepsilon^{3/2})$. However, the time scale $\mathcal{O}(1/\varepsilon^2)$ of validity in general can not be extended.

As a consequence of Theorem 4.1 modulating pulse solutions for the nonlinear wave equation (5) with the same carrier wave interact as predicted by the NLS equation, i.e., we have approximately the persistence of the modulating pulse solutions after the nonlinear interaction and $\mathcal{O}(1/\varepsilon)$ pulse-shifts in the nonlinear wave equation (5). For the transport of information through optical fibers the interaction of pulses is in general undesirable. However, even if the envelopes are in a very general form, like in real world technical devices, the NLS equation can now be used to compute numerically how far the modulating pulse solutions have to be separated such that there is no nonlinear interaction during the journey through the fiber.

5 Interaction of pulses with different frequencies

The information rate through the fiber can be increased by using different bands, i.e., different basic wave numbers, cf. [Ace00]. As explained in Remark 5.1 below (after fixing some notation), there is a simple argument why wave packets with different wave numbers do not interact in lowest order w.r.t. ε. Moreover, for pulses from Lemma 3.1 the argument can be refined, and in this section we explain that there is almost no interaction of such pulses associated to different carrier waves by giving an $\mathcal{O}(\varepsilon)$-bound for the possible shift of the envelope resulting from the interaction. For general wave packets the shift of the envelope will be in general $\mathcal{O}(1)$. Thus, it is advantageous to use well-prepared pulses for the transport of information.

We introduce subscripts A and B to indicate the wave numbers $k_A \neq k_B$ of each pulse, the associated group velocities $c_{g,A}$ and $c_{g,B}$, the envelope shifts x_A and x_B and so on. If the two pulses are separated initially, and, say, $x_A > x_B$ and $k_A < k_B$ such that $c_{g,A} < c_{g,B}$ and the faster pulse is in front, then, since the pulses are exponentially localized, it is natural to expect that the dynamics of the two pulses can be described by the sum of the two individual pulses, at least on the natural $\mathcal{O}(1/\varepsilon^2)$ time-scale. However, if the two pulses are, say, $\mathcal{O}(1/\varepsilon)$ separated initially, with $x_A > x_B$ and $k_A > k_B$, then, since the group velocities differ by $\mathcal{O}(1)$, the two pulses must interact on an $\mathcal{O}(1/\varepsilon^2)$ time-scale. Clearly this is the mathematically more interesting case.

For notational simplicity we assume that $\phi_A = \phi_B = 0$ and thus study the interaction of

$$\varepsilon v_{k_A}(x - c_{g,A}t + x_A, k_A(x - c_{p,A}t)) \quad \text{and} \quad \varepsilon v_{k_B}(x - c_{g,B}t + x_B, k_B(x - c_{p,B}t)), \quad k_A \neq k_B.$$

We prove that the form of the pulses is almost preserved and that the interaction mainly leads to phase-shifts $\varepsilon\Omega_A$ and $\varepsilon\Omega_B$ with $\Omega_A, \Omega_B \in \mathbb{R}$ bounded independent of ε.

Remark 5.1 That the amplitude equations for εv_{k_A} and εv_{k_B} decouple in lowest order can be seen as follows. Going into the scaling of the envelope, εv_{k_A} and εv_{k_B} have an amplitude and a width of order $\mathcal{O}(1)$. But since the group velocities differ by an order $1/\varepsilon$ in this scaling the interaction time of εv_{k_A} and εv_{k_B} is only $\mathcal{O}(\varepsilon)$. Thus, the influence of a term $v_{k_A} v_{k_B}$ on the dynamics of v_{k_A} and v_{k_B} is $\mathcal{O}(\varepsilon)$ in the NLS scaling and therefore in lowest order the evolution equations for v_{k_A} and v_{k_B} decouple. This argument is not restricted to v_{k_A} and v_{k_B}. It holds for all wave-packets. Moreover, this property can be observed in a number of problems. For modulating pulse solutions such a statement can be found for instance in [PW96] where it has been shown that the two NLS equations for counter-propagating waves decouple.

The estimates from [PW96] still only transfer into $\mathcal{O}(1)$-bounds for the possible envelope shifts of the pulses for $\varepsilon \to 0$. However, for well-prepared pulses, i.e., $n \geq 5$ in Lemma 3.1, by extracting explicitly the phase shift of the underlying carrier wave we can refine the bound on the envelope shifts to $\mathcal{O}(\varepsilon)$. In detail, we show that after interaction the solution is close to

$$\begin{aligned} &\varepsilon v_{k_A}(x - c_{g,A}t + x_A, k_A(x - c_{p,A}t) + \varepsilon\Omega_A(\eta_B)) \\ &\quad + \varepsilon v_{k_B}(x - c_{g,B}t + x_B, k_B(x - c_{p,B}t) + \varepsilon\Omega_B(\eta_A)), \end{aligned} \tag{17}$$

with explicit functions Ω_A, Ω_B, given by

$$\Omega_A = \int_{-\infty}^{\eta_B} \frac{3|B_1|^2}{\omega_A(c_A - c_B)} d\tilde{\eta}_B + \Omega_A^0 + \mathcal{O}(\varepsilon^2 e^{-r|\eta_B|}), \quad \eta_B = \varepsilon(x + x_B - c_{g,B}t), \tag{18}$$

$$\Omega_B = \int_{-\infty}^{\eta_A} \frac{3|A_1|^2}{\omega_B(c_B - c_A)} d\tilde{\eta}_A + \Omega_B^0 + \mathcal{O}(\varepsilon^2 e^{-r|\eta_B|}), \quad \eta_A = \varepsilon(x + x_A - c_{g,A}t), \tag{19}$$

where B_1 and A_1 are given by (9) with constants $C_{1,B}, C_{2,B}$ and $C_{1,A}, C_{2,A}$, respectively, and where Ω_A^0 and Ω_B^0 are constants which normalize the initial phases, see Fig. 5. Note that Ω_A depends on $x - c_{g,B}t$ and Ω_B on $x - c_{g,A}t$ as the phase shift accounts for so-called cross phase modulation.

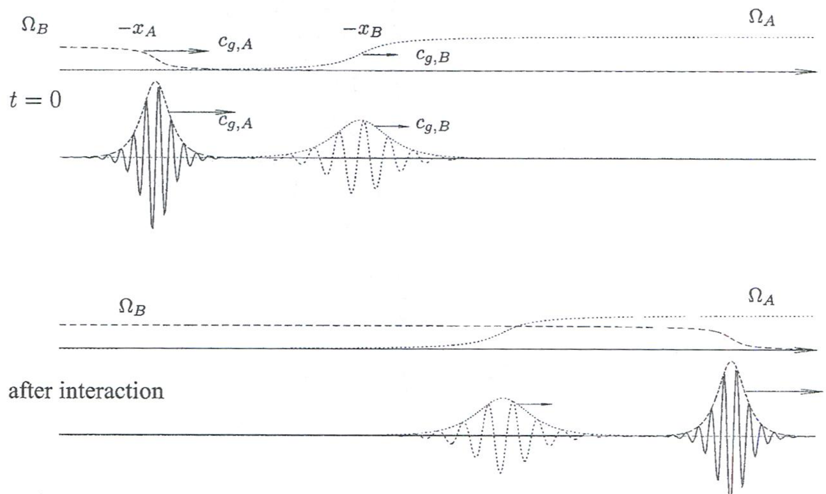

Figure 5: Illustration of the interaction of two pulses εv_{k_A} and εv_{k_B} with the associated cross-phase modulatins Ω_A and Ω_B. Here $k_a > k_B$ and the slower pulse is in front. Thus, $c_B - c_A < 0$ in (19) and Ω_B is a decaying function of x. The constants Ω_A^0 and Ω_B^0 have been chosen in such a way that at $t = 0$ (upper two pictures) there are no phase-shift for the pulses, i.e., Ω_B is exponentially small near the position x_B of εv_{k_B}, while Ω_A is exponentially small near the position $-x_A$ of ϵv_{k_A}. Note that Ω_B travels with ϵv_{k_A} and Ω_A with ϵv_{k_B}.

Theorem 5.2 *Let $s \geq 2$, $k_A, k_B > 0$, $k_A \neq k_B$, $\gamma_{1,A}, \gamma_{1,B} < 0$, $x_A, x_B \in \mathbb{R}$ in Lemma 3.1, and $T_0 > 0$. There exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist solutions u of (5) such that*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(x,t)-\varepsilon v_{k_A}(x-c_{g,A}t+x_A, k_A(x-c_{p,A}t)+\varepsilon\Omega_A(\eta_B)) \\ - \varepsilon v_{k_B}(x-c_{g,B}t+x_B, k_B(x-c_{p,B}t+\varepsilon\Omega_B(\eta_A))\|_{C_b^{s-1}} \leq C_2\varepsilon^3 \tag{20}$$

with v_{k_A}, v_{k_B} *from Lemma 3.1 and* Ω_A, Ω_B *given by (18),(19).*

Remark 5.3 To obtain an estimate for the physically relevant shift of the envelope, suppose that the error comes from a shift of the envelope. Then, due to the long wave form of the envelope, "vertical" estimates of order $\mathcal{O}(\varepsilon^3)$ in L^∞ can lead on a pulse of amplitude $\mathcal{O}(\varepsilon)$ only to a possible envelope shift εa of order $\mathcal{O}(\varepsilon)$, due to

$$\varepsilon g(\varepsilon(x+\varepsilon a)) - \varepsilon g(\varepsilon x) = \varepsilon g'(\varepsilon x)\varepsilon^2 a + \mathcal{O}(\varepsilon(\varepsilon^2 a)^2) = \mathcal{O}(\varepsilon^3).$$

Idea of the proof of Theorem 5.2 (See [CBSU06] for more details.) We make the ansatz

$$u(x,t) = \varepsilon\psi(x,t) := \varepsilon A_1(\eta_A)E + \varepsilon B_1(\eta_B)F \\ + \varepsilon^3 A_2(\eta_A, T)E + \varepsilon^3 B_2(\eta_B, T)F + \text{c.c.} + \text{h.o.t.} \tag{21}$$

where $T = \varepsilon^2 t$, and where

$$E = \mathrm{e}^{\mathrm{i}(k_A x - \omega_A t + \varepsilon\Omega_A(\eta_B))}, \quad F = \mathrm{e}^{\mathrm{i}(k_B x - \omega_B t + \varepsilon\Omega_B(\eta_A))}, \quad \eta_A = \varepsilon(x - c_{g,A}t), \ \eta_B = \varepsilon(x - c_{g,B}t).$$

In (21), h.o.t. stands for terms of higher order in ε, which are algebraically determined similar to A_3 in (11), and which do not lead to new aspects compared to Section 3. We choose A_1 and B_1 as given by Lemma 3.1. If we choose Ω_A, Ω_B to satisfy

$$\partial_{\eta_B}\Omega_A = \frac{3|B_1|^2}{\omega_A(c_{g,A} - c_{g,B})} \quad \text{and} \quad \partial_{\eta_A}\Omega_B = \frac{3|A_1|^2}{\omega_B(c_{g,B} - c_{g,A})} \tag{22}$$

which yields (18) and (19), then the coefficients at $\varepsilon^3 E$ and $\varepsilon^3 F$ vanish. At $\varepsilon^5 E$ and $\varepsilon^5 F$ we find that A_2, B_2 satisfy the linear equations

$$2i\omega_A\partial_T A_2 + (1 - c_{g,A}^2)\partial_{\eta_A}^2 A_2 + G_A = 0\,, \tag{23}$$

$$2i\omega_B\partial_T B_2 + (1 - c_{g,B}^2)\partial_{\eta_B}^2 B_2 + G_B = 0\,, \tag{24}$$

with, by construction, zero initial data, and where

$$G_A = 6|A_1|^2 A_2 + 3A_1^2 A_{-2} + 6(B_1 B_{-2} + B_2 B_{-1})A_1 \\ + \varepsilon^{-1}[\mathrm{i}(1 - c_{g,A}^2)\partial_{\eta_B}^2\Omega_A A_1 + 2\mathrm{i}(c_{g,A}c_{g,B} - 1)(\partial_{\eta_B}\Omega_A)(\partial_{\eta_A}A_1))],$$

$$G_B = 6|B_1|^2 B_2 + 3B_1^2 B_{-2} + 6(A_1 A_{-2} + A_2 A_{-1})B_1 \\ + \varepsilon^{-1}[\mathrm{i}(1 - c_{g,B}^2)\partial_{\eta_A}^2\Omega_B B_1 + 2\mathrm{i}(c_{g,A}c_{g,B} - 1)(\partial_{\eta_A}\Omega_B)(\partial_{\eta_B}B_1))].$$

The argument given in Remark 5.1 applied to the terms multiplied by ε^{-1} shows

Lemma 5.4 *There exists a* $C > 0$ *such that for all* $\varepsilon \in (0,1]$ *there exists a unique solution* $(A_2, B_2) \in C([0,T_0], H^s \times H^s)$ *to (23)-(24) with zero initial data. It satisfies*

$$\sup_{T\in[0,T_0]} \|(A_2, B_2)(T)\|_{H^s\times H^s} \leq C.$$

This shows that $\|\mathrm{Res}(\varepsilon\psi)\|_{H^s} = \mathcal{O}(\varepsilon^{11/2})$. Similar to the proof of Theorem 4.1 we write $u(x,t) = \varepsilon\psi(x,t) + \varepsilon^{7/2}R$, where we can employ the higher weight of R due to the smaller residual. The equation for R looks exactly as (14). Thus, $\sup_{t\in[0,T_0/\varepsilon^2]} \|R\|_{H^s} \leq C$ as above, which concludes the proof of Theorem 5.2. ∎

Theorem 5.2 can be extended in at least two directions. On a time scale $\mathcal{O}(1/\varepsilon^2)$ a modulating pulse can pass at most $\mathcal{O}(1/\varepsilon)$ many modulating pulses of width $\mathcal{O}(1/\varepsilon)$. The interaction of such a modulating pulse with $\mathcal{O}(1/\varepsilon)$ many modulating pulses with a different carrier wave can lead at most to an $\mathcal{O}(1)$-pulse shift. Thus, with respect to the question of the transport of information through optical fibers the influence of different frequencies to the dynamics at some frequency is negligible. Finally, a possibility to increase the rate of information through the fibers is to decrease the gap between the wave numbers. Formally we find for $k_A - k_B = \mathcal{O}(\varepsilon^\mu)$ with $0 \leq \mu \leq 1$ a pulse shift of order $\mathcal{O}(\varepsilon^{1-2\mu})$. Thus we must expect a certain payoff between the number of different carrier frequencies $1/(k_A - k_B)$ and the spacing of bits.

6 Pulses in photonic crystals: Standing light

One of the major goals of nanotechnology is photonics, i.e. the construction of 'electronic' devices where the electrons are completely replaced by photons. In this context, the question of optical storage plays a major role. One theoretical possibility are photonic crystals. These are optical materials with a periodic structure with a wave length comparable to the wave length of light. Due to the periodic structure the linearized problem is no longer solved by Fourier modes, but by so-called Bloch modes. The curves of eigenvalues plotted as a function over the Bloch wave numbers can now possess horizontal tangencies, i.e. vanishing group velocities. Thus, in principle, standing light pulses are possible. This will be explained in detail in the following, see also [BSTU06] for more details.

Again we consider a semilinear wave equation

$$\partial_t^2 u(x,t) = \chi_1(x)\partial_x^2 u(x,t) - \chi_2(x)u(x,t) - \chi_3(x)u^3(x,t) \tag{25}$$

with $x \in \mathbb{R}$ and $t \in \mathbb{R}$, $u = u(x,t) \in \mathbb{R}$, but now in a spatially periodic medium. This means that the coefficient functions $\chi_j = \chi_j(x)$ satisfy $\chi_j(x) = \chi_j(x+L)$ for $j = 1,2,3$. We assume here that the χ_j are smooth functions, that $\chi_1(x) > 0$ and that $\chi_2(x) > 0$ for all $x \in [0,L)$, and, without loss of generality, $L = 2\pi$ throughout this section. The linearized problem

$$\partial_t^2 v(x,t) = \chi_1(x)\partial_x^2 v(x,t) - \chi_2(x)v(x,t)$$

is solved by the Bloch waves

$$v(x,t) = \tilde{v}_n(\ell,x)\mathrm{e}^{\mathrm{i}\ell x}\mathrm{e}^{\mathrm{i}\omega_n(\ell)t}$$

where $n \in \mathbb{Z} \setminus \{0\}$, $\ell \in (-1/2, 1/2]$, with $\tilde{v}_n$ and ω_n determined by (27) below. Here, $\omega_n(\ell) \in \mathbb{R}$ satisfies $\omega_{n+1}(\ell) \geq \omega_n(\ell)$, $\omega_{-n}(\ell) = -\omega_n(\ell)$, and $\tilde{v}_n(x,\ell)$ satisfies

$$\tilde{v}_n(\ell, x) = \tilde{v}_n(\ell, x+2\pi) \quad \text{and} \quad \tilde{v}_n(\ell, x) = \tilde{v}_n(\ell+1, x)e^{ix}. \tag{26}$$

The Bloch wave transform of a function $u : \mathbb{R} \to \mathbb{C}$ is a generalization of Fourier transform and formally given by

$$\tilde{u}(\ell, x) = \sum_{j \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}jx} \hat{u}(\ell + j).$$

By construction, $\tilde{u}$ satisfies (26), and $\ell \in (-1/2, 1/2]$ is called a Bloch or pseudo wave number. From Parseval's identity $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$ it follows that Bloch transform is an isomorphism from $H^m(\mathbb{R}, \mathbb{C})$ to the Bloch space $L^2((-1/2, 1/2], H^m_{\mathrm{per}}((0, 2\pi)))$, and its inverse is given by

$$u(x) = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}\ell x} \tilde{u}(\ell, x)\, d\ell.$$

See [RS78, Sca99] for further properties and applications of Bloch transform.

For fixed Bloch wave number ℓ the Bloch modes $\tilde{v}_n(\ell, x)$ satisfy the spatially periodic eigenvalue problem

$$-\tilde{\Lambda}(\ell, \partial_x)\tilde{v}_n(\ell, \cdot) = \chi_1(\cdot)(\partial_x + \mathrm{i}\ell)^2 \tilde{v}_n(\ell, \cdot) - \chi_2(\cdot)\tilde{v}_n(\ell, \cdot) = -(\omega_n(\ell))^2 \tilde{v}_n(\ell, \cdot). \tag{27}$$

Since the operator $\tilde{\Lambda}(\ell, \partial_x)$ is elliptic in the bounded domain $[0, 2\pi)$ with periodic boundary conditions we have for fixed ℓ countable many eigenvalues $\lambda_n = \omega_n^2$, $n \in \mathbb{N}$. In the space $L^2_{\chi_1}(0, 2\pi)$ where

$$\langle \tilde{u}(\ell, \cdot), \tilde{v}(\ell, \cdot)\rangle_{\chi_1} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(\ell, x)\overline{\tilde{v}(\ell, x)} \frac{1}{\chi_1(x)} dx, \tag{28}$$

the operator $\tilde{\Lambda}(\ell, \partial_x)$ is positive definite and self adjoint such that the eigenvalues $\lambda_n(\ell)$ are real and positive. They are ordered by $\lambda_n(\ell) \leq \lambda_{n+1}(\ell)$.

We now explain the possibility of horizontal tangencies for the curves $\ell \mapsto \omega_n(\ell)$ by discussing periodic coefficients as perturbation of the spatially homogeneous case.

Example 6.1 The solutions of the constant coefficient case

$$\partial_t^2 v(x,t) = \partial_x^2 v(x,t) - v(x,t) \tag{29}$$

are given by the Fourier modes $v(x,t) = \mathrm{e}^{\mathrm{i}(kx \pm \mu(k)t)}$, where $(\mu(k))^2 = k^2 + 1$. We consider artificially the problem in a spatially periodic set-up. In a Bloch wave representation we have

$$v(x,t) = \mathrm{e}^{\mathrm{i}nx} \mathrm{e}^{\mathrm{i}\ell x} \mathrm{e}^{\mathrm{i}\omega_n^{\pm}(\ell)t},$$

where $k = n + \ell$, with $n \in \mathbb{Z}$ here and $\ell \in (-\frac{1}{2}, \frac{1}{2}]$. The eigenvalues are related by $\omega_n^{\pm}(\ell) = \pm\mu(n+\ell)$, i.e., they are obtained from wrapping $\pm\mu(\cdot)$ around a cylinder, see the left panel of Fig. 6.

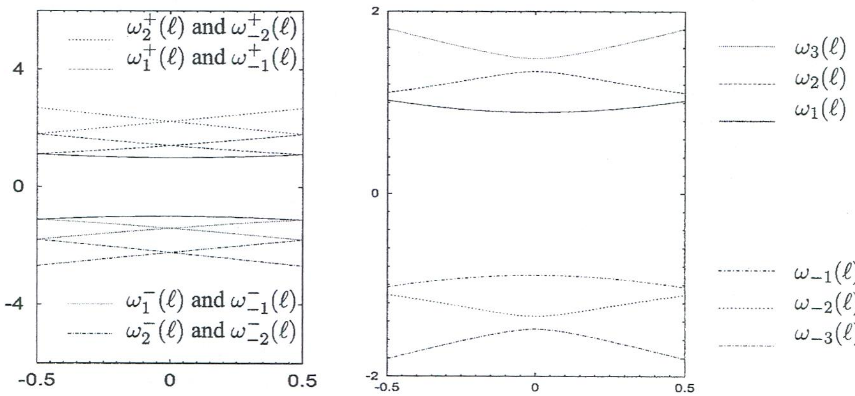

Figure 6: The curves of eigenvalues for the homogeneous case (29) in Bloch-representation, and the splitting of eigenvalues for (30). The Bloch modes of the ansatz (31) are strongly concentrated on an $\mathcal{O}(\varepsilon)$ neighborhood of the basic Bloch wave numbers $\pm\ell_0$ and the evolution of the wave packet will be strongly determined by the associated curves $\omega_{\pm n_0}$ at $\pm\ell_0$. Thus, the occurrence of horizontal tangencies as explained in Example 6.2 corresponds to vanishing group velocity c_g', i.e. to standing light pulses.

For all $\ell \in (-1/2, 1/2]$ except for $\ell = 0, 1/2$ all eigenvalues of $\tilde{\Lambda}(\ell, \partial_x)$ in Example 6.1 are simple. By classical perturbation arguments [Kat66], for periodic $\chi_j = 1 + \mathcal{O}(\delta)$ the eigenvalues are smooth functions of δ and stay separated for $\delta > 0$ sufficiently small. However, for $\ell = 0, 1/2$ all eigenvalues are double and generically for small $\delta > 0$ the eigenvalues will split. This is exactly what happens in the spatially periodic case.

Example 6.2 Let $\chi_2(x) = 1 + 2\delta\cos(2nx) = 1 + \delta(e^{i2nx} + e^{-i2nx})$ with $\delta > 0$ small and a fixed $n \in \mathbb{N}$. Setting

$$\tilde{v}_n(\ell, x) = \sum_{k\in\mathbb{Z}} \hat{v}_k^n(\ell) e^{ikx},$$

the eigenvalue problem (27) is given by the infinitely many equations

$$(1 + (k+\ell)^2)\hat{v}_k^n(\ell) + \delta(\hat{v}_{k+2n}^n(\ell) + \hat{v}_{k-2n}^n(\ell)) - \lambda_n(\ell)\hat{v}_k^n(\ell) = 0, \qquad (k \in \mathbb{Z}). \tag{30}$$

For $\delta = 0$ we have (with some abuse of notation) $\lambda_n(0) = \lambda_{-n}(0)$, i.e. a crossing of the curves of eigenvalues at $\ell = 0$. Due to the continuity of single eigenvalues or subspaces to eigenvalues separated from the rest, for small $\delta > 0$ and $\ell = 0$, the infinite dimensional eigenvalue problem in lowest order can be reduced to the two-dimensional problem

$$\det\begin{pmatrix} 1 + (-n)^2 - \lambda_n(0) & \delta \\ \delta & 1 + (n)^2 - \lambda_n(0) \end{pmatrix} = 0,$$

for $\hat{v}_n^n$ and $\hat{v}_{-n}^n$. Hence $\lambda_{\pm n}(0) = 1 + n^2 \pm \delta$. Thus, $\lambda_n(\ell)$ and $\lambda_{-n}(\ell)$ split at the crossings, i.e. at $\ell = 0$, and recombine in a different way. These new curves are also denoted with $\lambda_n(\ell)$ now ordered such that $\lambda_{n+1}(\ell) \geq \lambda_n(\ell)$ but now and in the following indexed with $n \in \mathbb{N}$. As before we let $\lambda_n(\ell) = \omega_n^2(\ell)$ and $\omega_n(\ell) = -\omega_{-n}(\ell) > 0$, see the right panel of Fig. 6.

Thus, on a linear level we have a situation as in the spatially homogeneous case: we have curves of eigenvalues over wave numbers except that associated eigenfunctions are no longer Fourier modes, but Bloch modes. Then, similar to the spatially homogeneous case, slow modulations in time and space of such a Bloch mode (indexed with n_0) may be described by the ansatz

$$u(x,t) = \varepsilon A(\varepsilon(x - c_g' t), \varepsilon^2 t)\tilde{v}_{n_0}(\ell_0, x)e^{i\ell_0 x}e^{i\omega_{n_0}(\ell_0)t} + \text{c.c.}, \tag{31}$$

where c.c. means complex conjugate, where $0 < \varepsilon \ll 1$ is a small parameter, and where $c_g' = \partial_\ell \omega_{n_0}(\ell_0)$ is the linear group velocity. The complex valued amplitude $A(X,T) \in \mathbb{C}$ describes slow modulations in time $T = \varepsilon^2 t$, and space $X = \varepsilon(x - c_g' t)$, of the underlying wave $\tilde{v}_{n_0}(\ell_0, x)e^{i\ell_0 x}e^{i\omega_{n_0}(\ell_0)t}$. The Bloch modes of the ansatz are strongly concentrated in an $\mathcal{O}(\varepsilon)$ neighborhood of the basic Bloch wave numbers $\pm\ell_0$ and the evolution of the wave packet will be strongly determined by the associated curves $\omega_{\pm n_0}$ at $\pm\ell_0$. Plugging the ansatz into (25) one finds that A has to satisfy a NLS equation

$$\partial_T A = i\nu_1 \partial_X^2 A + i\nu_2 A|A|^2 \tag{32}$$

with coefficients $\nu_1 = -\frac{1}{2}\partial_\ell^2 \omega_{n_0}(\ell_0) \in \mathbb{R}$ and

$$\nu_2 = \frac{3}{2\omega_{n_0}(\ell_0)} \int_0^{2\pi} \frac{\chi_3(x)}{\chi_1(x)} |v_{n_0}(\ell_0, x)|^4 \, dx \in \mathbb{R}.$$

The occurrence of the nonlinear term $i\nu_2 A|A|^2$ is a priori not clear at all. However, the nonlinear interaction corresponds in Bloch space to a convolution with respect to the Bloch wave numbers. Thus, the concentration of modes is respected by the nonlinear interaction which can be described in lowest order by $i\nu_2 A|A|^2$.

In general, the dispersion relation $\ell \mapsto \omega_n(\ell)$ and hence the coefficient ν_1 as well as ν_2 have to be calculated numerically. On the other hand, for a given material, these coefficients can be tailored by adjusting the grating, i.e. the periodic functions χ_j. This is a highly nontrivial optimization problem [HFBW01].

The justification of (32) for (25) in the sense of error estimates proceeds similar to the proof of Theorem 4.1, but the functional analysis becomes somewhat more complicated [BSTU06]. The physical detection of the pulses predicted by (32) is a nontrivial task, since they are localized in the photonic crystal and cannot be 'seen'. One possibility would be the interaction with other modulating pulses. However, similar to the analysis in §5, only pulses with carrier waves close to the carrier wave of the standing pulse will have any relevant, in terms of ε, effect on the standing pulse, and vice versa. Nevertheless, due to the higher dispersion, the influence is in general much larger than in homogeneous optical fibers, cf. [TPB04].

7 Single pulses II

We found approximate modulating pulse solutions with the help of the NLS equation up to a time-scale of order $\mathcal{O}(1/\varepsilon^2)$. Since these solutions are essential for the transport of information the following question occurs: do these solutions exist for all $t \in \mathbb{R}$?

More precisely, are there 'breather solutions', which are time-periodic solutions in a moving frame and which are spatially localized, i.e., which decay to zero for $|x| \to \infty$? Such solutions are known explicitly for the sine-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - \sin(u),$$

which first appeared in differential geometry in the description of surfaces with constant negative curvature [Enn70], but which also appears in crystallography and in particle physics. In fact, like the NLS equation, the sine-Gordon equation is a completely integrable Hamiltonian system. See [DJ89] for more background and references.

Thus, the question is whether 'breathers' can also exist in other nonlinear wave equations, for instance of the type

$$\partial_t^2 u = \partial_x^2 u - u + g(u),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth, odd function which satisfies $g(u) = \mathcal{O}(u^3)$ and $g'''(0) > 0$. It turns out that for $g(u)$ close to $u - \sin(u)$ the sine-Gordon equation is the only such equation. For a precise statement see [BMW94, Den93]. In the following we explain why this 'non existence of breathers' result holds. Moreover, we will explain positive results for generalized breather solutions.

The solutions we are interested in are obtained from the ansatz

$$u(x,t) = v(x - c_{\mathrm{g}} t, x - c_{\mathrm{p}} t) = v(\xi, y),$$

where v is periodic in y with period $2\pi/k_0$ for some $k_0 > 0$. They are homoclinic solutions of the evolutionary system

$$(1 - c_{\mathrm{g}}^2)\partial_\xi^2 v + (1 - c_{\mathrm{p}}^2)\partial_y^2 v - v + g(v) = 0, \tag{33}$$

which generalizes the spatial dynamics approach of Kirchgässner [Kir82], i.e., we look for v with

$$\lim_{\xi \to \pm\infty} v(\xi, y) = 0.$$

In order to obtain (33) we have chosen $c_g = 1/c_p$ according to the linear relation $c_g' = 1/c_p'$.

Hence, v has to be in the intersection of the stable and unstable manifold of the origin. The stable and unstable manifolds are the nonlinear counterparts to the stable and unstable subspaces in case of linear equations and are tangential to these subspaces. Therefore, we look at the linearization around the fixed point $(v, \partial_\xi v) \equiv (0,0)$ in order to compute the dimensions of these manifolds. The linearization of (33) is given by

$$(1 - c_{\mathrm{g}}^2)\partial_\xi^2 v + (1 - c_{\mathrm{p}}^2)\partial_y^2 v - v = 0. \tag{34}$$

Since we are interested in periodic solutions w.r.t. y we use Fourier series

$$v(\xi, y) = \sum_{m \in \mathbb{Z}} v_m(\xi) e^{imk_0 y}$$

and find $\partial_\xi^2 v_m = -\lambda_m^2 v_m$ which is solved by $u_m(x) = e^{i\lambda_m x} u_m(0)$ where $\lambda_m^2 = \frac{m^2 k_0^2 (1 - c_{\mathrm{p}}^2) + 1}{(1 - c_{\mathrm{g}}^2)}$.

Due to the cubic nonlinearity we can restrict to odd $m \in \mathbb{Z}$. Therefore, for c_p close to c_p' the eigenvalues λ_m are on the imaginary axis for $|m| > 3$. The eigenvalues $\lambda_{\pm 1}$ are on the real axis for $c_p < c_p'$. Hence we have a two-dimensional stable and a two-dimensional unstable manifold. These manifolds intersect for the sine-Gordon equation, but in general two two-dimensional manifolds will not intersect in an infinite-dimensional phase space. This makes the sine-Gordon equation exceptional in this class of equations.

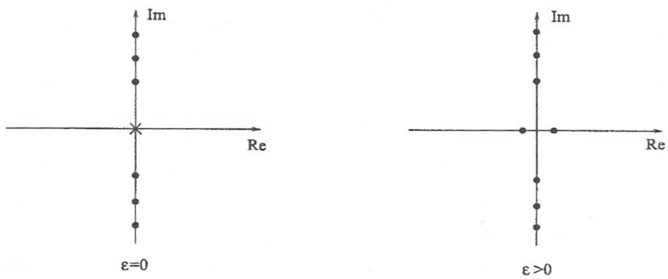

Figure 7: The spectrum of the linearization (34), where ε is defined in Theorem 7.1.

A time-periodic solution in a moving frame is called generalized moving breather or generalized modulating pulse solution if not necessarily

$$\lim_{\xi \to \pm\infty} v(\xi, y) = 0,$$

but $v(\xi, y)$ is small for $|\xi| \to \infty$. In [GS01], the existence of generalized modulating pulse solutions with $\mathcal{O}(\varepsilon^n)$-tails has been established. For simplicity we restrict to $g(u) = u^3$.

Theorem 7.1 *Fix a positive integer n and a positive real number k_0. For sufficiently small $\varepsilon > 0$ (depending upon n and k_0) there exists an infinite-dimensional, continuous family of modulating pulse solutions to equation (5) of the form*

$$u(x, t) = v(x - c_g t, x - c_p t),$$

where v is $2\pi/k_0$-periodic in its second argument and $c_p = c_p' + \gamma_1 \varepsilon^2$, $c_g = 1/c_p$. These solutions satisfy

$$v(\xi, y) = v(-\xi, y), \quad |v(\xi, y) - 2h(\xi, y, \varepsilon)| \leq \varepsilon^{n+1}, \qquad \xi, y \in \mathbb{R},$$

where $h(\xi, y, \varepsilon) = \varepsilon B_{\text{pulse}}(\varepsilon\xi) \sin k_0 y + \text{c.c.} + \mathcal{O}(\varepsilon^2)$ and $\lim_{\xi \to \pm\infty} h(\xi, y, \varepsilon) = 0$.

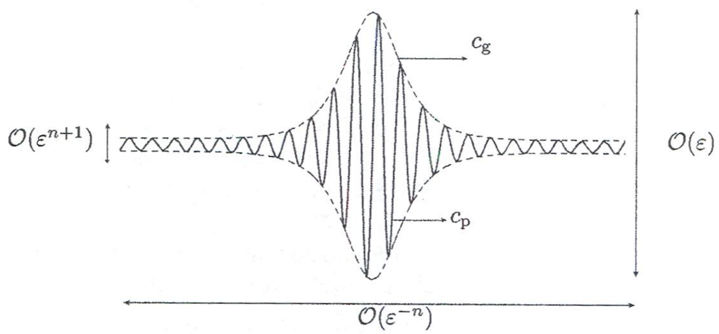

Figure 8: A generalized modulating pulse solution.

The modulating pulse solutions of Theorem 7.1 are found in the intersection of the infinite-dimensional center stable and infinite-dimensional center unstable manifold. For $|\xi| \to \infty$ the solutions converge with some exponential rate towards the center manifold. Thus, a secular growth of the solutions is possible. However, for this special equation the boundedness for $|\xi| \to \infty$ follows with the help of the Hamiltonian structure due to the fact that the Hamiltonian restricted to the center manifold is positive definite.

For general, especially quasilinear, systems the norm induced by the Hamiltonian is too weak compared with the norm used for the construction of the invariant manifolds. Thus, in general, generalized modulating pulse solutions can only be constructed for $|\xi| \leq 1/\varepsilon^n$, cf. [GS05]. This result has been improved in [GS06] to exponentially small tails and exponentially large intervals, i.e., $|\xi| \leq \exp(-1/\varepsilon)$.

8 Outlook and related fields

The above analysis can be extended into a number of directions. First we may consider different constitutive laws for the polarization, as for instance

$$\partial_t^2 p(x,t) = u(x,t) + \partial_t^2 (u(x,t)^3)$$

leading to quasilinear systems, cf. [GS05].

Recently so-called ultra-short pulses have attracted a lot of interest, cf. [SW04]. They play an important role in spectroscopy. For such pulses the length of the envelope and the wavelength of the underlying carrier wave have a comparable size.

In materials with broken up-down symmetry also quadratic terms are present. Then, from a mathematical point of view, the proof of the above approximation results is a much more challenging task. The idea is to use normal form transforms or averaging methods to eliminate the quadratic terms and to reduce the proof to the cubic case, cf. [Sch98, BSTU06]. The case of quadratic resonant media has been treated recently in [Sch05].

There is another famous system with dispersive behavior for which the NLS equation can be derived, namely the water wave problem, cf. [Zak68]. Estimates for the residual can be found in [CSS92]. Here, quadratic terms are present. The elimination of these terms is complicated due to some resonance at the wavenumber $k = 0$ and other resonances present in case of small positive surface tension. Estimates for model problems can be found in [DS05]. A first attempt for the water wave problem as been made in [SW06] where the validity of the approximation over at least the right time scale has been shown.

More generally, as already pointed out in the introduction, the methods reviewed here can be applied to all dispersive nonlinear equations for which the NLS equation can be derived.

There are still many open questions. A serious difficulty in the description of photonic crystals comes from the fact that the coefficient functions χ_j very often are step functions, i.e., they are not smooth. Another challenging problem is the justification of the NLS equation when the original equation possesses quasilinear quadratic terms. The elimination of these terms by normal form transforms gives a loss of regularity complicating the local existence and uniqueness theory of solutions substantially.

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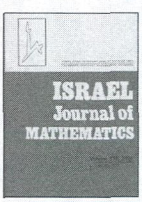

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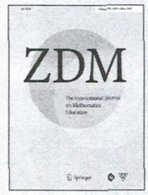

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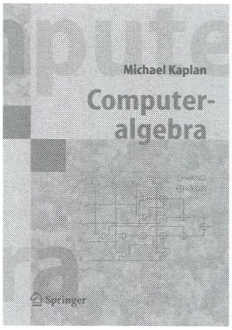


M. Kaplan
Computeralgebra

Berlin u.a., Springer, 2004, 391 S., € 39,95

Das vorliegende Buch ist ein Lehrbuch, das aus Vorlesungen des Autors für Informatiker und Mathematiker an der TU München hervorgegangen ist. In 8 Kapiteln und 2 Anhängen wird eine Auswahl von grundlegenden Algorithmen aus verschiedenen Bereichen der Computeralgebra vorgestellt.

Der Autor geht wie folgt vor. Die ersten vier Kapitel, die fast die Hälfte des Buches darstellen, sind Grundlagen gewidmet. Kapitel 1 ist eine kurze Einleitung, in der insbesondere andere Quellen zur Einführung in die Computeralgebra sowie vergleichende Literatur zu Computeralgebrasystemen aufgelistet werden. In Kapitel 2 werden grundlegende Themen wie Algorithmen und ihre Komplexität, Resultanten oder Mignotte-Schranken angesprochen. Kapitel 3 ist dem Rechnen mit homomorphen Bildern (chinesischer Restsatz und Hensel-Liftung) gewidmet. Kapitel 4 ist das mit Abstand längste Kapitel des Buches. Behandelt werden die Darstellung und Arithmetik von Zahlen und Polynomen.

In Kapitel 5 werden Algorithmen zur Faktorisierung ganzer Zahlen vorgestellt (Pollard-ρ, Pollard-$(p-1)$, Lenstras Elliptische-Kurven-Methode, Kettenbruch-Methode von Morrison-Brillhart, quadratisches Zahlensieb). Kapitel 6 beschäftigt sich mit den Anfängen der univariaten Polynomfaktorisierung (Algorithmen von Berlekamp, Cantor-Zassenhaus und Zassenhaus). Gegenstand von Kapitel 7 ist die unbestimmte Summation. Vorgestellt werden die Algorithmen von Moenck und Gosper. Kapitel 8 ist eine sehr kurze Einführung in das Konzept der Gröbner-Basen (Algorithmus von Buchberger). Die Anhänge lieferen einige Informationen zu ausgewählten Computeralgebrasystemen.

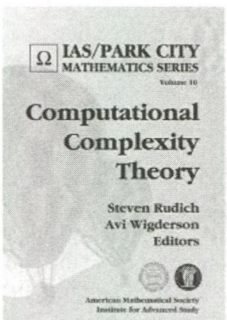
Das Buch vermittelt einen ersten Eindruck von verschiedenen Bereichen der Computeralgebra. Die behandelten Algorithmen werden in Pseudocode dargestellt, analysiert und durch zahlreiche, gut ausgewählte Beispiele illustriert. Es sind gerade die Beispiele, die vorlesungsbegleitend oder beim Selbststudium nützlich sind.

Zum tieferen Verständnis der Materie müssen jedoch auch andere Quellen herangezogen werden. So fehlt etwa die schnelle Fourier-Transformation im Kapitel über Arithmetik. Es werden Algorithmen zur Faktorisierung ganzer Zahlen vorgestellt, nicht aber Primzahltests. Im Kapitel über Polynomfaktorisierung werden die 1965–1970 entworfenen Algorithmen zur univariaten Faktorisierung beschrieben (diese sind auch heute noch von zentraler Bedeutung). Neuere Entwicklungen zur univariaten Faktorisierung, die Faktorisierung über Zahlkörpern, modulare gcd-Algorithmen, multivariate Polynomfaktorisierung oder absolute Polynomfaktorisierung werden aber nicht angesprochen. Die fundamentale Bedeutung von Buchbergers Algorithmus zur Berechnung von Gröbnerbasen wird nicht klar genug herausgearbeitet. Einige Bemerkungen zu Querverbindungen zwischen den einzelnen Kapiteln (etwa Polynomfaktorisierung + Gröbner-Basen $\Longrightarrow$ Zerlegung von Varietäten oder die klassische Anwendung von Resultanten zur Elimination) sowie ausführlicher besprochenene Beispiele für praktische Anwendungen würden das Buch gut ergänzen.

Etwas irritiert hat mich der Klappentext. Das dort angedeutete Vorhaben, an Beispielen die Leistungsfähigkeit von Computeralgebrasystemen aufzuzeigen, wird weder qualitativ noch quantitativ in Angriff genommen.

Zusammenfassend kann gesagt werden, dass das Buch eine erste Einführung in die Computeralgebra darstellt, die durch die behandelten Beispiele besticht. Es ist deswegen eine empfehlenswerte Ergänzung der vorhandenen Literatur.

Saarbrücken W. Decker



S. Rudich and A. Widgerson
Computational Complexity Theory
IAS/Park City Mathem. Series, Vol. 10

American Math. Soc., 2004, 389 S., $ 69,–

Seit Beginn der 90er Jahre widmet sich das Park City Mathematics Institute in Zusammenarbeit mit dem Institute for Advanced Study in Princeton der Förderung der mathematischen Ausbildung und Forschung mit einem breiten Spektrum an Veranstaltungen für Lehrer, Studenten, Dozenten und Forscher. In jedem Jahr wird ein Schwerpunktthema gewählt, das dann auch Gegenstand einer dreiwöchigen Graduate Summer School ist, deren Teilnehmer, mehrheitlich Doktoranden und Postdoktoranden und Forscher aus anderen Bereichen, von den Grundlagen des jeweiligen Gebiets bis zu aktuellen Themen der Forschung geführt werden sollen. Im Sommer des Jahres 2000 war mit *Computational Complexity Theory erstmals* ein Thema aus der Theoretischen Informatik Schwerpunkt. Der vorliegende Band gibt (mit einer Ausnahme) die von den Dozenten, die ohne Einschränkung zu den führenden Köpfen der Szene gerechnet werden dürfen, gehaltenen Vorlesungen wieder. An der Fülle des Materials und den technischen Ansprüchen erkennt man klar den „Steilkurs"-Charakter dieser Veranstaltung. Neben motivierenden Erläuterungen zu den Themen und Begriffen, Definitionen und Resultaten, die der Orientierung in der Landschaft der Komplexitätstheorie dienen, geben die Beiträge mit ihren ausgeführten Beweisen und Beweisskizzen einen guten Eindruck von der Arbeitsweise in diesem Gebiet. Inhaltlich kann der Band für sich in Anspruch nehmen, in konzentrierter Form eine Art Bestandsaufnahme der Komplexitätstheorie (oder doch wenigstens von wichtigen Teilen davon), etwa 30 Jahre nach den Arbeiten von Stephen Cook, Leonid Levin und Richard Karp, zu geben, mit denen dieses Forschungsgebiet begründet wurde.

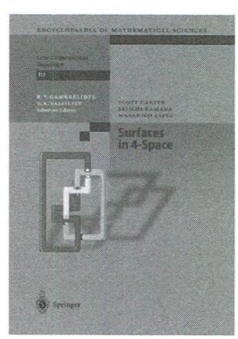
Die einführenden acht kompakten Vorlesungen *Complexity Theory: From Gödel to Feynman* von Steven Rudich stecken den Rahmen dieser Sommerschule ab. Ausgehend von den grundlegenden Begriffsbildungen um das notorische P-vs.-NP-Problem, über Komplexitätsmaße, Komplexitätsklassen, Reduktionen und Relativierungen bis hin zu unteren Schranken für Schaltkreise und im Rahmen der Beweiskomplexität, wird ein Panorama vorgeführt, das in vielen Punkten in den folgenden Beiträgen vertieft behandelt wird. Der Titel dieser Vorlesungen ist allerdings etwas irreführend: zwar wird Kurt Gödels berühmter Brief an John von Neumann aus dem Jahr 1956 abgedruckt (im originalen Deutsch und in englischer Übersetzung) und kurz diskutiert, aber über Richard Feynman und seine Ideen erfährt man (auch später) so gut wir nichts. Ergänzt werden diese einführenden Vorlesungen durch eine sehr knappe, aber instruktive Darstellung der *Average Case Complexity* von Avi Widgerson und drei Vorlesungen *Exploring Complexity through Reductions* von Sanjeev Arora, wo das wichtige Thema der *Probabilistically Checkable Proofs* und der Nicht-Approximierbarkeit der exakten Lösungen von manchen NP-schwierigen Problemen schon einmal behandelt wird. Dazu wird mit dem Beweis von Todas Theorem zu Zählproblemen und polynomieller Hierachie auch ein echter „Klassiker" der Komplexitätstheorie geboten. Die folgenden drei Vorlesungen von Ran Raz über *Quantum Computing* beschränken sich auf eine elementare Einführung in den Formalismus und ein Darstellung von Shor's polynomiellen Faktorisierungsalgorithmus – sicher ein bahnbrechendes Resultat, das man aber inzwischen ähnlich dargeboten auch in vielen Lehrbüchern findet, was man von den meisten anderen Beiträgen so nicht sagen kann. Andere Bereiche des Quantencomputing und der Quanteninformation werden nicht angesprochen.

Die zweite Woche der Sommerschule, mit der Überschrift *Lower Bounds*, hatte drei Themen: *Circuit and Communication Complexity* (Ran Raz), *Proof Complexity* (Paul Beame) und *Arithmetic and Algebraic Complexity* (Michael Ben-Or). An den beiden erstgenannten Beiträgen erkennt man, wie unterschiedlich die Dozenten an ihre Aufgabe herangegangen sind: Ran Raz gibt eine exemplarische und – inklusive Beweise – leicht nachvollziehbare Einführung in sein Thema. Paul Beame geht sein Thema sehr systematisch an, was bei die Vielzahl von formalisierten logischen Beweiskonzepten die Übersicht erleichtert. Aber daher können viele der in dieses umfassende Bild gehörenden Resultate nur ohne Beweis zitiert werden. Zum letzten der drei genannten Thema fehlt bedauerlicherweise eine Ausarbeitung in dem vorliegenden Band – dieses umfangreiche und wichtige Teilgebiet der Komplexitätstheorie ist somit hier überhaupt nicht vertreten.

Der große Themenbereich *Randomness in Computation* bestimmte die dritte Woche, wobei zwei Schwerpunkte auszumachen sind: einmal die Untersuchung von Pseudozufallsgeneratoren und Pseudozufallsfunktionen, sowie der Technik des *Derandomisierens*, die in den beiden Beiträgen *Pseudorandomness I* und *II* von Oded Golddreich bzw. Luca Trevisan dargestellt sind; sodann die faszinierenden Konzepte für interaktives, probabilistisches Beweisen (*Interactive Proofs, Zero-Knowledge Proofs, Probabilistically Checkable Proofs*) mit ihren erstaunlichen Resultaten, von Salil Vadha und Madhu Sudan kompetent präsentiert. Angesichts der Kompliziertheit der Methoden und Resultate, die zu den PCP-Charakterisierungen der Komplexitätsklasse NP führen, darf man in diesem Rahmen natürlich keine lehrbuchartige komplette Aufarbeitung erwarten, aber eine wertvolle Wegleitung aus erster Hand ist es allemal.

Ein solcher Band mit Beträgen verschiedener Autoren (teilweise wohl auch auf studentischen Mitschriften beruhend) hat – wie angedeutet – fast zwangsläufig Unausgewogenheiten. Der Vorzug, in vielen Bereichen von ausgewiesenen Experten einen authentischen Eindruck von der Denk- und Arbeitsweise in deren Spezialgebieten anhand von grundlegenden und aktuellen Resultaten zu erhalten, ist greifbar. Aber ein durchstrukturiertes Lehrbuch hält man nicht in der Hand, eher eine Sammlung von interessanten, anregenden, unterschiedlich konzipiertenTutorials auf durchwegs hohem Niveau, das den *state of the art* nach dreissig Jahren Komplexitätstheorie widerspiegelt.

Erlangen V. Strehl



S. Carter,
S. Kamada,
M. Saito
Surfaces in 4-Space
Enz. Math. Sciences
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Berlin u. a., Springer, 2004, 213 S., € 84,95

This book provides a survey of knotted surfaces in Euclidian 4-space including their recent results on quandle cocycle invariants and its applications. 2-dimensional knot theory is a higher version of classical 1-dimensional knot theory, but it has very different properties from the classical case.

In Chapter 1 of the book they discuss basic methods of drawing pictures of knotted surfaces, such as motion pictures, normal forms, broken surface diagrams, and braid chart descriptions. These methods are used in the area, and the pictures illustrated in this book are very beautiful.

Chapter 2 is devoted to the known methods of constructing knotted surfaces. In particular, deform-spun knots and ribbon knots play important rolls of the study. They also explain how to construct a compact 3-manifold, whose boundary is a given knotted surface, by the Seifert algorithm.

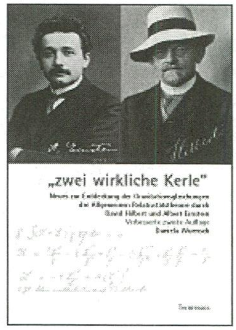
In Chapter 3 they review the invariants of knotted surfaces derived from the exterior of a knotted surface in the space. The Wirtinger presentation of the fundamental group of an exterior is given in various ways by using the descriptions in Chapter 1. They also explain the Faber-Levine pairing and its generalization due to Kawauchi.

Chapter 4 contains their most recent, interesting works on quandles. They develop quandle homology and cohomology theory and define the invariants by using quandle cocycles in a state-sum form. The quandle cocycle invariants of knotted surfaces are natural generalizations of the Dijkgraaf-Witten invariant for 3-manifolds. They also have many applications, for example, a lower bound of triple point numbers and non-invertibility of knotted surfaces.

The appendix is devoted to the tables of quandles with at most six elements and their homology groups due to Uegaki, which are useful to calculating the cocycle invariants of knotted surfaces.

This book is intended for graduate students and mathematicians who are studying low dimensional topology.

Chiba (Japan) S. Satoh



D. Wuensch
Zwei wirkliche Kerle: Neues zur Entdeckung der Gravitationsgleichungen der Allgemeinen Relativitätstheorie durch Albert Einstein und David Hilbert

Termessos, Göttingen, 2005, 120 S., € 24,95

In seiner *Unsichtbaren Loge* schreibt Jean Paul „Die Anstrengung der empfindenden Phantasie ist unter allen geistigen die entnervendste; ein Algebraist überlebt allemal einen Tragödiensteller." Lichtenberg vermutete aus ähnlichen Gründen, dass Mathematik zur Verlängerung des Lebens beitragen könne. Mit anderen Worten, die rationale, adrenalinarme Tätigkeit des Mathematikers macht zuweilen Mühe, ist aber sicher gesund!

Nun hat es gerade in den letzten Jahren eine Entwicklung, wenn auch nicht in der Mathematik selbst, so doch in der Geschichte der Mathematik gegeben, die den Adrenalinspiegel – zumindest bei den Beteiligten – in die Höhe trieb. Das Buch von Frau Wuensch enthält eine präzise Darstellung und Bestandsaufnahme dieses Falles, logisch, sachlich, frei von Emotionen und trotzdem spannend wie ein Kriminalroman.

Zur Sache: Im Jahre 1915 hatte Albert Einstein mit seiner bekannten Arbeit *Zur allgemeinen Relativitätstheorie* die Physik durch eine neue aufregende Sichtweise bereichert, die seiner speziellen Relativitätsheorie einen umfassenden einheitlichen Rahmen gab. Im gleichen Jahre reichte David Hilbert eine Arbeit *Die Grundlagen der Physik* zum gleichen Thema ein. Beide Arbeiten wurden nahezu zeitgleich eingereicht, die Hilbertsche Publikation erschien jedoch erst im Jahre 1916. Die beiden Autoren standen vor den Veröffentlichungen in lebhaftem Kontakt miteinander, das heißt, die beiden Arbeiten sind nicht unabhängig entstanden. Nach bisheriger Meinung hat Albert Einstein von David Hilbert bei der Formulierung der mathematischen Grundlagen der allgemeinen Relativitätstheorie profitiert.

Im Jahre 1997 erschien in der Zeitschrift *Science* ein Artikel von Leo Corry, Jürgen Renn und John Stachel, in dem die sensationelle Behauptung aufgestellt wurde, dass umgekehrt Hilbert von Einstein profitiert habe. Das Hauptargument dieser Autoren war, dass in den von dem ersten Autor in der Staats- und Universitätsbibliothek Göttingen aufgefundenen Fahnenkorrekturen von Hilberts Arbeit eine Formel fehle, die in der endgültigen Arbeit zu finden sei. Hilbert habe zwar die Einsteinschen Gravitationsgleichungen in der impliziten Form aufgeführt, jedoch in den Fahnenkorrekturen daraus nicht den zugehörigen Variationsterm hergeleitet. Diese sensationelle Nachricht provozierte eine Anzahl von Reaktionen und Gegenreaktionen, nicht nur in wissenschaftshistorischen Zeitschriften, auch in der *Washington Post*, der *Neuen Zürcher Zeitung* und natürlich auch im Internet. Eine Website mit recht kräftigen Anwürfen in ausgesprochen unakademischer Wortwahl musste inzwischen nach Einschreiten eines Anwalts zurückgezogen werden.

Im Jahre 2004 hat Friedwardt Winterberg eine nicht minder aufregende Entdeckung publiziert: Hilberts Fahnenkorrekturen sind offenbar nicht mehr im Originalzustand, irgend jemand hat sie auseinandergerissen und ausgerechnet die Stelle herausgeschnitten, an der sich die fragliche Formel befinden müsste. Dieser Tatbestand wird von Corry, Renn und Stachel auch nicht bestritten. Wir haben hier also die absurde Situation, dass die „Anklage“ gegen Hilbert auf die Nichtexistenz einer Formel begründet ist, die sich auf einem Teil der Fahnenkorrekturen befunden hätte, der ebenfalls nichtexistent ist. Ein solcher, zweifach auf Nichtexistenz gegründeter Beweis erweckt Unbehagen (Schliemann hat in Troja keine Drähte ausgegraben, also besaßen die Trojaner drahtlose Telegraphie!).

In dem Buch von Frau Wuensch wird in umfassender und minutiöser Weise Spurensicherung betrieben. Sie hat das Umfeld der beiden Kontrahenten zur Tatzeit gründlich untersucht, und man begegnet hier guten Bekannten wie Felix Klein, Walter Baade, Max Born und anderen. Sie hat sich intensiv mit den in den Archiven noch vorhandenen Manuskripten Hilberts auseinandergesetzt, und sie hat überzeugend versucht, die Vorgänge, die zu dem jetzigen Zustand der Fahnenkorrekturen geführt haben, zu analysieren. Dieser Teil des Buches liest sich spannend wie ein Kriminalroman. Die Autorin hat am „Tatort“ in der Staats- und Universitätsbibliothek Göttingen akribisch recherchiert und zahlreiche spannende Details zutage gebracht, wie die Faltung der Fahnenkorrekturen oder die Art und Weise der Ausschneidung, aber sie befasst sich auch mit der Persönlichkeit Hilberts: Welche Motive hätte er gehabt, sein Manuskript zu verstümmeln, wie hielt er es generell mit Ausschneidungen? Dies geht so weit, dass die Autorin abschätzt, welche Größe der verschwundene Ausschnitt gehabt hatte und wieviel Inhalt sich auf einer Fläche dieser Größe unterbringen lässt. Sie hat die Manuskripte Hilberts statistisch untersucht, um zum Beispiel festzustellen, auf welche Weise Hilbert Manuskripte bezifferte. Sie hat die Handschrift Hilberts analysiert und sie ist der Frage nachgegangen, warum die Fahnenkorrekturen auf die nun vorliegende Weise gefaltet sind. Ihre Schlussfolgerung ist eindeutig und nachvollziehbar: Die Manipulationen an den Fahnenkorrekturen sind in neuerer Zeit – genauer nach 1985 – entstanden.

Neben diesen materiellen Indizien gibt es natürlich auch noch den *modus operandi* der Beteiligten. Ist es einsichtig, dass David Hilbert, der die „Weiterführung der Methoden der Variationsrechnung“ im Jahre 1900 als 23. Problem gestellt hatte, der auf dem Gebiet der Variationsrechnung und der Invariantentheorie Grundlegendes geleistet hatte, eine – zumindest für ihn – relativ naheliegende Konsequenz, die er in seinem publizierten Artikel mit den Worten „. . .wie leicht ohne Rechnung ... folgt ..." erst gesehen haben soll, nachdem er die inzwischen erschienene Arbeit Einsteins in Händen hatte? Auch diesen innermathematischen Fragen geht Frau Wuensch mit der ihr eigenen Gründlichkeit und Sauberkeit nach. Dabei steht sie nicht allein mit dieser Vorgehensweise, zum Beispiel haben sich auch A. A. Logunov, M. A. Mestvirishvili und V. A. Petrov in einem unlängst in den *Phys. Uspekhi* erschienenen Artikel in die Debatte eingeschaltet. In einer sehr klaren und gründlichen physikalischen Argumentation kommen sie zu dem Resultat „... the conclusions drawn in these papers [Corry, Renn, Stachel] are completely groundless."

Es ist bezeichnend, dass die beiden *dramatis personae* selbst das hier angesprochene Problem bemerkenswert gelassen hinnahmen, was die eingangs zitierte Theorie von der besonderen Art der Konfliktbewältigung unter Mathematikern zu stützen scheint. Eine anfängliche Verstimmung – Hilbert äußerte sich ungehalten darüber, dass Einstein ihn in seiner Arbeit nicht zitiert hatte – wurde alsbald beigelegt. Weitaus wichtiger ist – und dies wird von Frau Wuensch besonders hervorgehoben – dass hier Unvergleichbares verglichen wird. Einstein war Physiker, Hilbert Mathematiker. Das eigentliche Anliegen Hilberts in seiner Arbeit ist der Versuch einer Axiomatisierung der Physik (6. Hilbertsches Problem: Mathematische Behandlung der Axiome der Physik). Die Gleichung, die heute die Gemüter so erregt, war sicherlich für Hilbert eine naheliegende Konsequenz, steht aber eher am Rande der Hilbertschen Überlegungen. Für Einstein hingegen war die Mathematik ein wichtiges Werkzeug, aber eben nur ein Werkzeug, wobei er nach Felix Klein „mehr von einem starken physikalischen Instinkt als von klarer mathematischer Einsicht gleitet wird." So sehen es auch Logunov, Mestvirishvili und Petrov, die bezüglich Einsteins Darstellung auf „intuitive considerations" hinweisen und darauf, dass die Gleichungen nicht immer „hergeleitet" (derived) sondern „geraten" (guessed) seien.

Frau Wuensch gibt der Versuchung nicht nach, aus ihren Untersuchungen ihrerseits eine neue Anklage zu formulieren, auf eine Person zu zeigen und auszurufen „Thou art the man!" Es bleibt dem Leser überlassen, seine eigenen Schlüsse zu ziehen. Nach den eingangs erwähnten Emotionen wirkt sie sachliche Kühle dieses – nichtsdestoweniger ungemein spannenden – Buches sehr wohltuend.

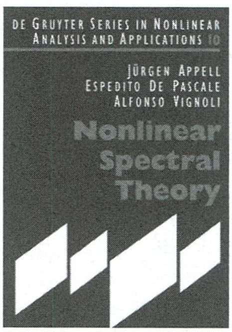
Man kann das Buch aus verschiedenen Gründen empfehlen. Zum einen ist es einfach eine fesselnde Lektüre, darüber hinaus liefert es einen interessanten Einblick in das wissenschaftliche Umfeld von Einstein und Hilbert und zeigt den mühevollen Weg bis zur endgültigen Formulierung der allgemeinen Relativitätstheorie. Das Buch zeigt auch auf eine – wie ich meine – mustergültige Art, wie mathematikhistorische Forschung sein sollte: Man muss die Fakten mit Akribie und wissenschaftlicher Ehrlichkeit recherchieren, und erst wenn diese nur noch eine einzige Schlussfolgerung zwingend zulassen, dann sollte man diese ziehen, auch wenn die Versuchung, ein sensationelles Ergebnis vorschnell zu publizieren, manchmal verlockend ist. Wie Brecht es im *Leben des Galilei* fordert:

> Ja, wir werden alles, alles noch einmal in Frage stellen. Und wir werden nicht mit Siebenmeilenstiefeln vorwärtsgehen, sondern im Schneckentempo. Und was wir heute finden, werden wir morgen von der Tafel streichen und erst wieder anschreiben, wenn wir es noch einmal gefunden haben. Und was wir zu finden wünschen, das werden wir, gefunden, mit besonderem Misstrauen ansehen. ... Sollte uns dann aber jede andere Annahme als diese unter den Händen zerronnen sein, dann keine Gnade mehr mit denen, die nicht geforscht haben und doch reden.

Mein Resumee über das Buch lässt sich am besten in Einsteins Worten formulieren: „Lieber Leser! Resümiert wäre profaniert. Selber lesen!" Ich würde sogar so weit gehen wollen, mit Lichtenberg zu empfehlen: „Wer zwei Paar Hosen hat, mache eins zu Geld und schaffe sich dieses Buch an."

Man darf auf das Buch der gleichen Autorin über Theodor Kaluza gespannt sein.

Hamburg U. Eckhardt



J. Appell, E. De Pascale, A. Vignoli
Nonlinear Spectral Theory
de Gruyter Ser. in Nonlin. Anal. and Appl. 10

Berlin u. a., de Gruyter, 2004, 408 S., € 148,–

This is a unique book in the mathematical literature which systematically deals with the nonlinear spectral theory, looks for the common points with the classical linear spectral theory on one hand and points out the striking differences on the other hand.

Nonlinear spectral theory is relatively new field of mathematics. Its origin falls within the second half of the twenties century. It is far from being complete and many fundamental questions remain to be open here. The main focus of this book is therefore formulated by the authors in the following question: "How should we define a spectrum for nonlinear operators which attempts to preserve the useful properties of the linear case, but admits applications to a possibly large variety of nonlinear problems?"

Since the spectral theory for linear operators appeared to be one of the most important topics of functional analysis and one of the most important tools in solving linear operator equations, it is natural to look for analogies of the notion "spectrum" in the nonlinear theory.

It was always tacitly assumed that a reasonable definition of a spectrum of a continuous nonlinear operator should satisfy some minimal requirements, namely:

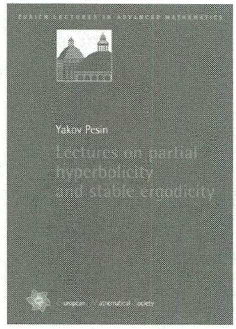
- it should reduce to the familiar spectrum in case of linear operators
- it should share some of the usual properties with the linear spectrum (e. g. compactness)
- it should contain the eigenvalues of the operator involved
- it should have nontrivial applications, i.e. those which may not be obtained by other known means.

As the authors show, in contrast to the linear case, the spectrum of a nonlinear operator contains practically no information on this operator. The authors finally convince the reader that it is not the intrinsic structure of the spectrum itself which leads to interesting applications, but its property of being a useful tool for solving nonlinear equations. The reason consists in the fact that the authors are more interested in studying specific nonlinear equations rather than abstract spectra.

This book is an excellent presentation of the "state-of-the-art" of contemporary nonlinear spectral theory as well as the glimpse of the diversity of the directions in which current research is moving. The whole text consists of 12 chapters. The authors recall the basic facts obout the spectrum of a bounded linear operator in a Banach space in the first chapter. In Chapter 2 some numerical characteristics which provide a quantitative description of certain mapping properties of nonlinear operators are studied. The classical Kuratowski measure of noncompactness plays the key role here. Chapter 3 is devoted to the general invertibility results. In particular, the conditions which guarantee that the local invertibility of a nonlinear operator implies its global invertibility are of interest. The Rhodins and the Neuberger spectra are studied in Chapter 4. In Chapter 5 the authors study a spectrum for Lipschitz continuous operators first proposed by Kachurovskij in 1969 and a spectrum for linearly bounded operators introduced recently by Dörfner. Chapter 6 is concerned with a spectrum for certain special continuous operators introduced by Furi, Martelli and Vignoli in 1978 and its modification introduced recently by Appell, Giorgieri and Väth. The Feng spectrum is discussed in detail in Chapter 7. Chapter 8 is devoted to the study of "local spectrum" due to Väth, which is called in the literature as "phantom". In Chapter 9 the authors investigate the modification of Feng spectrum proposed by Feng and Webb and another spectra introduced by Singhof-Weyer, Weberand Infante-Webb. Chapter 10 is devoted to the study of nonlinear eigenvalue problems. The authors focus on the notion of a "nonlinear eigenvalue", nonlinear analogue of the Krein-Rutman theorem, connected eigenvalues, etc. In Chapter 11 they show how numerical ranges may be used to localize the spectrum of a nonlinear operator on the real line or in the complex plane. Selected applications are presented in the last Chapter 12.

The exposition of nonlinear spectral theory in this book is self-contained. All major statements are proved, each definition and notion is carefully illustrated by examples. To understand this text does not require any special knowledge and only modest background of nonlinear analysis and operator theory is required. Due to these facts this book can be used also as a "textbook" for graduate students who are looking for new and interesting problems. Especially, those who are interested in the solvability and bifurcation of nonlinear operator equations which represent nonlinear boundary value problems or nonlinear integral equations will find a lot of inspiration there. The book is addressed not only to mathematicians working in analysis but also to non-specialists who want to get on idea of the development of spectral theory for nonlinear operators in the last 30 years. The bibliography is rather exhaustive and so this text will certainly serve as an excellent reference book for many years. This volume should not be missing at any department of mathematics and in any mathematical library.

Pilsen — P. Drábek



Y. Pesin
Lectures on partial hyperbolicity and stable ergodicity
Zürich Lec. in Adv. Math.

Zürich, European Math. Soc., 2004, 122 S., € 28,–

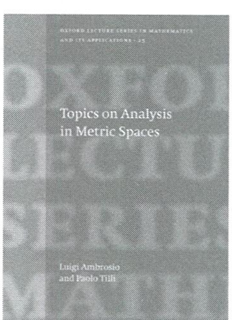
Das vorliegende Buch ist eine Einführung in die Theorie der partiell hyperbolischen Systeme, einem sich in letzter Zeit schnell entwickelten Zweig der Theorie der glatten dynamischen Systeme. Der Autor zählt zu den weltweit führenden Experten auf diesem Gebiet und ist einer der Begründer dieser modernen Theorie. Obwohl dieses Gebiet seit etwa 30 Jahren erforscht wird, sind gerade in den letzten Jahren sehr viele Publikationen erschienen. Das Buch von Pesin kann aber als das erste umfassende und einführende Lehrbuch angesehen werden. Es ist sowohl für fortgeschrittene und interessierte Stundenten und Doktoranden als auch für Wissenschaftler, die sich mit dynamischen Systemen beschäftigen äußerst empfehlenswert.

Das Buch beginnt mit der grundlegenden Theorie der hyperbolischen Systeme. Dabei wird besonders die Theorie der stabilen und instabilen Blätterungen beschrieben. Ausgehend von ihrer Existenz werden ihre weitergehenden Eigenschaften wie absolute Stetigkeit und Hölderstetigkeit diskutiert. Daran anknüpfend wird der Beweis der strukturellen Stabilität von Anosovsystemen skizziert.

Nach diesen Grundlagen der Theorie der hyperbolischen Systeme wird die Theorie der partiell hyperbolischen Systeme untersucht. Diese unterscheiden sich von den hyperbolischen durch das Auftreten „neutraler" Richtungen. Nach Beschreibung deren Eigenschaften wird das Hauptaugenmerk auf den Beweis des fundamentalen Satzes von Pugh und Shub über die stabile Ergodizität von partiell hyperbolischen Systemen gerichtet. Ein zentraler Punkt dabei ist die Untersuchung der stabilen Erreichbarkeit (stable accesibility).

Das Buch ist in einem hervorragendem Stil geschrieben. Die Beweise und Ideen sind klar strukturiert. Der Aufbau ist in sich schlüssig und logisch klar strukturiert. Die Notationen verzichten auf unnötige Kompliziertheit. Das alles trägt zu einer sehr guten Lesbarkeit bei. Obwohl ein gewisses Grundwissen über die Theorie der glatten dynamischen Systeme hilfreich sein kann, speziell wenn man an den vollständigen Beweisen der Grundlagen der hyperbolischen Theorie interessiert ist, ist dieses Buch in sich geschlossen und kommt weitgehend ohne zusätzliche Literatur aus.

Lund J. Schmeling



L. Ambrosio, P. Tilli
Topics on Analysis on Metric Spaces
Oxford Lect. Ser. in Math. and its Appl. 25

Oxford University Press, 2004, 133 S., £ 29,50

Bei der Behandlung von Problemen im Bereich Partieller Differentialgleichungen und Mathematischer Physik finden grundlegende Konzepte der Analysis und der Maßtheorie auf euklidischen Räumen Anwendung. In den letzten Jahren setzte sich jedoch die Erkenntnis durch, dass diese fundamentalen Konzepte verallgemeinert werden können, indem man die zugrunde liegenden euklidischen Räume durch geeignete Klassen von metrischen Räumen ersetzt, was neben einem tieferen Verständnis der zugrunde liegenden Strukturen auch zu eleganten und verständlichen Beweisen der Resultate führt.

Ziel des vorliegenden Buches ist eine elementare Einführung in die Grundlagen der Analysis auf metrischen Räumen. Das Buch selbst basiert auf einer Vorlesung des ersten Autors, die er an der Scuola Normale in Pisa im Jahre 1999 abgehalten hat.

Kapitel 1 gibt eine Einführung in die Maßtheorie. Dabei wird der Zugang von Carathéodory über äußere Maße entwickelt. Neben der grundlegenden Definition werden die wichtigsten Begiffe wie reguläre Maße, die Messbarkeit von Mengen bzgl. des Maßes und das Kriterium von Carathéodory für Messbarkeit vorgestellt. Danach wenden sich die Autoren den signierten und vektorwertigen Maßen zu. Insbesondere diskutieren sie den Rieszschen Darstellungssatz für beschränkte lineare Funktionale auf einem lokal-kompakten topologischen Hausdorff-Raum E mit Werten in $\mathbb{R}^n$ und skizzieren dessen Beweis. Schließlich wird das wichtige Konzept der schwachen Konvergenz von Maßen etabliert.

In Kapitel 2 führen die Autoren dann das k-dimensionale **Hausdorff-Mass** $\mathcal{H}^k$, $k \geq 0$, auf einem metrischen Raum (E, d) ein und studieren dessen wichtigste Eigenschaften. Derartige Maße stellen eine Verallgemeinerung der klassischen Konzepte von Länge, Flächeninhalt und Volumen dar. Auf den Spezialfall $(E, d) = (\mathbb{R}^n, |\cdot|)$, $k = n$, wird besonders eingegangen und gezeigt, dass das n-dimensionale Hausdorff-Maß auf $\mathbb{R}^n$ mit dem n-dimensionalen Lebesgue-Maß $\mathcal{L}^n$ übereinstimmt. Schließlich wird noch auf die für viele Anwendungen wichtige Frage eingegangen, ob ein gegebenes Radon-Maß μ durch ein Hausdorff-Maß darstellbar ist oder ob man zumindest die Hausdorff-Dimension des Trägers von μ kontrollieren kann. Dazu betrachtet man $\mu(B_r(x))/r^k$ im Limes $r \downarrow 0$ und wird auf die Definition der k-dimensionalen **oberen (unteren) sphärischen Dichte** bzw. der **sphärischen Dichte** geführt. Die Annahme an die k-dimensionale obere (untere) Dichte, dass sie $\geq t$ ($\leq t$), $t > 0$, auf $B \subset E$ ist, erlaubt die Abschätzung $\mu(B) \geq t\mathcal{H}^k(B)$ (bzw. $\leq t2^k\mathcal{H}^k(B)$). Die im Beweis benötigten Überdeckungssätze (etwa der Überdeckungssatz von Vitali), die den Schluss vom Lokalen zum Globalen ermöglichen, werden zuvor ausführlich diskutiert.

Das dritte Kapitel beschäftigt sich mit der Klasse der Lipschitz-stetigen Funktionen $\mathrm{Lip}(E,F)$ zwischen metrischen Räumen (E, d_E) und (F, d_F) und deren Verbindungen zum Hausdorff-Maß. Zunächst wird der Frage nachgegangen, inwieweit es möglich ist, Lipschitz-Funktionen, die auf einer Teilmenge A von E definiert sind, zu einer Lipschitz-Funktion auf E fortzusetzen. Im Fall $(F, d_F) = (\mathbb{R}, |\cdot|)$ wird dies durch den elementaren Satz von McShane geklärt. Allgemeinere Aussagen werden diskutiert, aber nicht bewiesen (etwa der Satz von Kirszbraun, Lipschitz-Funktionen zwischen metrischen Räumen mit Krümmungsschranken in Sinne von Alexandrov, etc.). Danach werden Anwendungen vorgestellt, die das Verhalten des Hausdoff Maßes unter Lipschitz-Funktionen betreffen. Unter anderem wird eine Vorstufe der **Koflächenformel** diskutiert. Genauer: Ist $\phi \in \mathrm{Lip}(E,F)$, $\mathcal{H}^k(E) < \infty$ und $m \leq k$, so lässt sich das Integral des $(k-m)$- dimesionalen Flächeninhaltes $\mathcal{H}^{k-m}(\phi^{-1}(y))$ der Niveaumengen $\phi^{-1}(y)$, $y \in F$, bzgl. $d\mathcal{H}^k(y)$ durch $c_{m,k}\,\mathrm{Lip}(\phi)^m\,\mathcal{H}^k(E)$ abschätzen.

Im Anschluss wird auf die wichtige Charakterisierung der Lipschitz-Funktionen $\mathrm{Lip}(\mathbb{R}^n, \mathbb{R})$, als genau derjenigen $L^1(\mathbb{R}^n)$-Funktionen mit Distributions-Gradient $\nabla u \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, und den Satz von Rademacher eingegangen. Letzterer garantiert, dass jede Lipschitz-Funktion fast überall (bzgl. des Lebesgue-Maßes) differenzierbar ist.

Zum Abschluss von Kapitel 3 wird die allgemeine Flächenformel für Lipschitz-Funktionen $\phi\colon \mathbb{R}^k \to \mathbb{R}^n$, $n \geq k$, vorgestellt und diskutiert. Auf den technisch aufwendigen Beweis wird verzichtet. Statt dessen wird die eindimensionale Flächenformel, d. h. $k = m = 1$, für die Klasse der fast überall differenzierbaren Funktionen $\phi\colon \mathbb{R} \to \mathbb{R}$, die Nullmengen in Nullmengen abbilden, vollständig bewiesen.

Das vierte Kapitel ist zunächst dem Begriff der **Geodätischen** in einem metrischen Raum (E,d) gewidmet. Für Kurven $\gamma\colon [0,1] \to E$ werden die Begriffe der totalen Variation und der Rektifizierbarkeit eingeführt. Dies erlaubt die parametrische Formulierung des Problems der kürzesten Verbindungskurve zwischen zwei Punkten $x, y \in E$ als

$$\min\left\{\mathrm{Var}(\gamma) : \begin{array}{c}\gamma \in \mathrm{Lip}([0,1],E),\\ \gamma(0)=x,\ \gamma(1)=y\end{array}\right\}.$$

Daneben hat man auch die intrinsische Formulierung:

$$\min\left\{\mathcal{H}^1(C) : \begin{array}{c}C \text{ abgeschlossen und}\\ \text{zusammenhängend}, x,\ y \in C\end{array}\right\}.$$

Das Ziel ist nun zu zeigen, dass sowohl die parametrische als auch die intrinsische Formulierung des Problems der kürzesten Verbindung eine Lösung besitzt und dass diese übereinstimmen. Dazu wird zunächst das Konzept der **metrischen Ableitung** eingeführt. Eine Kurve $\gamma\colon [a,b] \to E$ ist in $t \in (a,b)$ metrisch differenzierbar, falls der Grenzwert

$$|\dot{\gamma}|(t) := \lim_{h\to 0} \frac{d(\gamma(t+h), \gamma(t))}{h} \in \mathbb{R}$$

existiert. Für Lipschitz-Kurven $\gamma\colon [a,b] \to E$ zeigt sich dann, dass die metrische Ableitung in fast allen Punkten in $[a,b]$ existiert und dass die totale Variation $\mathrm{Var}(\gamma)$ gerade das Integral der metrischen Ableitung $|\dot{\gamma}|(t)$ über $[a,b]$ ist. Dies bedeutet insbesondere, dass für fast alle $t \in [a,b]$ gilt: $\dot{\ell}(t) = |\dot{\gamma}|(t)$, wobei $\ell(t) = \mathrm{Var}_a^t(\gamma)$. D. h. die metrische Ableitung ist für fast alle Punkte gerade die Änderungsrate der Länge der Kurve. Damit ist man dann in der Lage jede Lipschitz-Kurve $\gamma\colon [a,b] \to E$ nach der Bogenlänge umzuparametrisieren, d. h. man findet eine Lipschitz-Kurve $\tilde{\gamma}\colon [0,L] \to E$, $L := \mathrm{Var}(\gamma)$, mit metrischer Ableitung $|\dot{\tilde{\gamma}}| = 1$ fast überall auf $[0,L]$ und $\tilde{\gamma}([0,L]) = \gamma([a,b])$. Die Existenz parametrischer geodätischer Verbindungskurven ergibt sich dann wie im klassischen Riemannschen Fall, indem man eine Minimalfolge geeignet nach der Bogenlänge umparametrisiert und auf die umparametrisierten Kurven den Satz von Arzelà-Ascoli anwendet (dabei ist natürlich vorauszusetzen, dass die Klasse der Lipschitz-Kurven, die x mit y verbinden, nicht leer ist).

Der Beweis, dass die intrinsische Formulierung oder allgemeiner, das Variationsproblem

$$\min\left\{\mathcal{H}^1(C) : \begin{array}{c} C \text{ abgeschlossen,} \\ \text{zusammenhängend, } C' \subset C \end{array}\right\}$$

mit einer gegebenen abgeschlossenen nichtleeren Menge $C' \subset E$, eine Lösung besitzt, ist tiefliegender und benötigt Methoden der geometrischen Maßtheorie. So wird unter anderem benutzt, dass eine abgeschlossene, zusammenhängende Teilmege C eines vollständigen metrischen Raumes mit $\mathcal{H}^1(C) < \infty$ kompakt ist und je zwei Punkte $x, y \in C$ durch eine injektive rektifizierbare Kurve verbunden werden können. Desweiteren ist C abzählbar 1-rektifizierbar, d. h. es gibt höchstens abzählbar viele Lipschitz-Kurven $\gamma_i\colon [0,1] \to C$, sodass $\mathcal{H}^1(C \setminus \bigcup_{i=1}^{\infty} \gamma_i([0,1])) = 0$. Im Zusammenspiel mit Eigenschaften der **Hausdorff-Konvergenz** für abgeschlossene Mengen ergeben diese Strukturresultate einen Unterhalbstetigkeitssatz für das 1-dimensionale Hausdorff-Maß bzgl. Konvergenz im Hausdorf-Abstand; genauer gilt in einem vollständigen metrischen Raum: Ist C_j eine Folge nicht leerer abgeschlossener zusammenhängender Mengen, die im Hausdorff-Abstand gegen C konvergiert, so ist C zusammenhängend und es gilt

$$\mathcal{H}^1(C) \le \liminf_{j\to\infty} \mathcal{H}^1(C_j).$$

Mit Hilfe dieses Unterhalbstetigkeitsresultats lässt sich der Beweis, dass die intrinsische Formulierung des Problems der kürzesten Verbindung zwischen zwei Punkten in einem metrischen Raum eine Lösung besitzt, unter der Annahme, dass jede abgeschlossene Kugel in E kompakt ist, leicht führen. Darüber hinaus kann jede minimierende Menge C mit $\{x,y\} = C' \subset C$ durch eine injektive Lipschitz-Kurve parametrisiert werden. Dies bedeutet insbesondere, dass die intrinsische Formulierung äquivalent zur parametrischen Formulierung ist.

Der Abschluss von Kapitel 4 ist dem Gromov-Hausdorff-Abstand d_{GH} zwischen kompakten metrischen Räumen und dem damit verbundenen Konvergenzbegriff auf der Menge der Isometrieklassen kompakter metrischer Räume $\mathcal{M}$ gewidmet. Neben der grundlegenden Eigenschaft, dass $d_{\mathrm{GH}}\colon \mathcal{M} \times \mathcal{M} \to [0,\infty)$ eine vollständige Metrik auf $\mathcal{M}$ ist, wird der Zusammenhang zwischen Gromov-Hausdorff- und Hausdorff-Konvergenz studiert. Gezeigt wird u. a., dass das Konzept der Konvergenz im Gromov-Hausdorff-Abstand bis auf den Übergang zu einer Teilfolge (modulo einer isometrischen Einbettung) äquivalent zur Konvergenz im Hausdorff-Abstand ist (Gromovscher Einbettungssatz). Die Flexibilität des Konzeptes der Gromov-Hausdorff-Konvergenz wird durch den Beweis eines Existenzsatzes zum Steiner Problem in gewissen metrischen Räumen E untermauert.

Kapitel 5 ist dem Konzept der Sobolev-Räume auf metrischen Räumen gewidmet. Ausgehend von zwei metrischen Räumen (E, d_E) und (F, d_F), einem Borel Maß μ auf E, das auf beschränkten Mengen endlich ist, und $p \in [1,\infty]$ definiert man nach Hajasz: $u\colon E \to F$ gehört zur Sobolev-Klasse $W_m^{1,p}(E,\mu,F)$, falls es eine nicht-negative Funktion $g \in L^p(E,\mu)$ und eine μ-Nullmenge $N \subset E$ gibt, sodass für alle $x, y \in E \setminus N$ gilt

$$(1) \quad d_F(u(x),u(y)) \le d_E(x,y)(g(x)+g(y)).$$

Als unmittelbare Konsequenzen werden vorgestellt: Die Existenz eines minimalen g, d. h.

mit minimaler L^p-Norm, im Fall $1 < p < \infty$, eines Lipschitz-Approximationssatzes vom Lusin-Typ, die Charakterisierung der Sobolev-Funktionen $W_m^{1,p}(E,\mu,F)$ vermöge Lipschitz-Funktionen $\phi\colon F \to \mathbb{R}$ und schließlich im euklidischen Fall $E = \Omega \subset \mathbb{R}^n$, Ω offen, $\mu =$ das n-dimesionale Lebesgue Maß $\mathcal{L}^n$, die Folgerung, dass der Distributionsgradient einer Funktion aus $W_m^{1,p}(E,\mu,\mathbb{R})$ in $L^p(\Omega,\mathbb{R}^n)$ liegt. Insbesondere handelt es sich bei Funktionen $u \in W_m^{1,p}(\Omega,\mathcal{L}^n) \cap L^P$ um klassische Sobolev-Funktionen, d. h. $u \in W^{1,p}(\Omega)$.

Nach diesen eher elementaren Folgerungen wird das Konzept der Hardy-Littlewood Maximalfunktion im Kontext der metrischen Räume eingeführt. Dazu betrachtet man nur Borel-Maße μ, die eine **Verdopplungseingenschaft** besitzen, d. h. es existiert eine Konstante C, sodass gilt

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x)) \quad \forall\, x \in E,\ r > 0\,.$$

Für eine Borel Funktion $f\colon E \to [0,\infty]$ ist die Hardy-Littlewood Maximalfunktion dann definiert durch

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B_r(x)} \int_{B_r(x)} f(y)\, d\mu(y)\,.$$

Als wichtige Eigenschaften der Maximalfunktion werden die schwache L^1-Abschätzung und die Stetigkeit des Maximalfunktionenoperators

$$L^p(E,\mu) \ni f \mapsto Mf \in L^p(E,\mu),$$

$1 < p < \infty$, nachgewiesen.

Die Einführung der Hardy-Littlewood Maximalfunktion erlaubt dann im euklidischen Fall unter milden Voraussetzungen an Ω den Beweis der Äquivalenz: $W_m^{1,p}(\Omega,\mathcal{L}^n) \cap L^p(\Omega,\mathcal{L}^n) = W^{1,p}(\Omega)$. Dazu zeigt man, dass für $u \in W^{1,p}(\mathbb{R}^n)$ (1) mit $g = M(|\nabla u|)$ gilt. Dies ist auch die Motivation für die von Hajasz gegebene Defintion. Den Abschluss des Kapitels bilden Poincaré- und Sobolev-Ungleichungen und der Auswahlsatz von Rellich für Funktionen aus metrischen Sobolev Räumen.

Abgerundet wird das Buch durch eine kurze Einführung in die Integrationstheorie. Ausgangspunkt ist die Formel von Cavalieri, also $\int_X u\, d\mu = \int_0^\infty \mu(\{u > t\})\, dt$, die man zur Definition des Integrals macht. Dabei ist $\mu\colon 2^X \to [0,\infty]$ nicht fallend und $u\colon X \to [0,\infty]$. Dieser Zugang wurde von DeGiorgi im Jahre 1983 in einer Reihe von Vorlesungen an der Scuola Normale Superiore Pisa vorgestellt.

Das Buch von Ambrosio & Tilli stellt eine wunderschöne Einführung in die Grundlagen der Analysis auf metrischen Räumen dar. Es liest sich sehr kurzweilig. Die Beweise sind sehr gut verständlich und die Argumente hervorragend motiviert. Jedes Kapitel wird abgerundet durch eine Reihe von Aufgaben unterschiedlichen Schwierigkeitsgrades. Als Grundlage für eine Vorlesung oder ein Seminar (ca. drittes Studienjahr) ist das Buch hervorragend geeignet.

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ISSN 0012-0456